EXPLICIT CONSTRUCTION OF NON-LINEAR PSEUDO-ANOSOV MAPS, WITH NONMINIMAL INVARIANT FOLIATIONS

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ABSTRACT. Starting from any linear pseudo-Anosov map φ on a surface of genus $g \geqslant 2$, we construct explicitly a family of non-linear pseudo-Anosov maps f by adapting the construction of Smale's Derived from Anosov maps on the two-torus. This is done by perturbing φ at some fixed points. We first consider perturbations at every conical fixed point and then at regular fixed points. We establish the existence of a measure μ, supported by the non-trivial unique minimal component of the stable foliation of f, with respect to which f is mixing. In the process, we construct a uniquely ergodic generalized interval exchange transformation with a wandering interval that is semi-conjugated to a self-similar interval exchange transformation. This generalized interval exchange transformation is obtained as the Poincaré map of a flow renormalized by f. When f is $\mathcal{C}^2$, the flow and the generalized interval exchange transformation are $\mathcal{C}^1$.

1. INTRODUCTION

Along with the Spectral Decomposition of Diffeomorphisms theorem, Smale [26, Theorem I.6.2] introduced Derived from Anosov maps – build by perturbing a linear Anosov transformation of the two-torus at a fixed point – in order to provide an example of transformation with a proper hyperbolic nonwandering set. The topological classification of such admissible basic sets is performed in dimension 2 in [3, 2]. Later, Coudène [12, Chapter 9] introduced an explicit formula for such a map – which differs slightly from the one used in [20, Section 17.2]. This formula is interesting from several viewpoints: it is an explicit Axiom A diffeomorphism which renormalizes a $\mathcal{C}^1$ flow, a slightly more general setting than the recent Giulietti–Liverani horocyclic flows construction [17] – see [10] for more details. Furthermore, the Poincaré map of the flow on a transversal closed curve provides a Denjoy counter-example.

The purpose of this paper is to provide explicit generalizations of Coudène's construction to surfaces of higher genus.

Since the work of Thurston [14], it is known that pseudo-Anosov maps are relevant examples of surface diffeomorphisms. Essentially, in the orientable case, a pseudo-Anosov map on a surface S_g of genus $g \geqslant 2$ can be defined as an element of the homotopy class of a map preserving a flat metric on S_g and locally given – in the natural coordinates associated to the half-translation structure – by the action

Date: April 23, 2021.

This work was carried out during a one-year internship under the supervision of Corinna Ulcigrai. Research supported by Institut für Mathematik, Universität Zürich, Switzerland and by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 787304).

of a diagonal and hyperbolic matrix of determinant one. Furthermore, up to a sign change, the matrix is constant on $S_g \setminus \Sigma$ – where Σ denotes the set of conical points. The element of the homotopy class given by the action of the matrix is called *linear pseudo-Anosov*, and we call *non-linear pseudo-Anosov* any element of its homotopy class. It easily follows from this definition that iterates of a pseudo-Anosov map are also pseudo-Anosov maps. See for instance [21] for an equivalent definition and many instructive examples.

Many of the dynamical properties of linear pseudo-Anosov maps are known and can be found in [13, 14] and more recently in [15]. For instance, stable and unstable measured foliations are uniquely ergodic, every leaf of these foliations is dense, and the map is Bernoulli with respect to the product measure – which has full support. However, most of these properties do not hold for general *non-linear* pseudo-Anosov maps since homotopy classes do not generally coincide with conjugacy classes – for example, it is not the case for homeomorphism of the circle.

In this paper, we prove the following result using an explicit construction:

Theorem 1.1. *If φ is a linear pseudo-Anosov map on S_g preserving an Abelian differential which vanishes on the finite set Σ of conical points, then there exist an integer n and a non-linear pseudo-Anosov map f homotopic to φ^n, a proper f-invariant compact connected hyperbolic subset $K \subsetneq S_g \setminus \Sigma$, and a f-invariant measure μ supported by K such that*

(i) K is an Axiom A attractor for $f^{-1}|_{S_g \setminus \Sigma}$,
(ii) K is locally the product of an interval with a Cantor set,
(iii) K is the minimal set of the stable foliation of f,
(iv) f is mixing with respect to μ. In fact, μ is the unique SRB measure of f^{-1}.

In particular, since $K \neq S_g$, f and φ^n are homotopic but cannot be conjugated.

The transformation f is constructed in a similar fashion as in [12, Chapter 9] by perturbing φ^n at conical points. A similar construction can be performed by perturbing φ^n at a regular fixed point, and counterpart results are stated in Section 5 (Theorem 5.8).

In this particular construction, along the lines of the proof of Theorem 1.1, we are bound to consider a flow renormalized by f. This leads to our second result:

Theorem 1.2. *For every function f constructed as in Theorem 1.1, there exists a flow h_t on $S_g \setminus \Sigma$ such that*

(i) h_t is complete on K and $\frac{\mathrm{d}}{\mathrm{d}t} h_t(x)|_{t=0}$ spans the stable foliation of f, for all x in K,
(ii) for all real t and all x in $S_g \setminus \Sigma$, whenever both sides of the equation are well defined, $f \circ h_{\lambda t}(x) = h_t \circ f(x)$, where $\lambda > 1$ is the dilation of φ^n,
(iii) h_t is uniquely ergodic, with unique invariant measure supported by K,
(iv) K is an attractor (for future and past times) for h_t, on which the flow acts minimally.

Furthermore, if f is $\mathcal{C}^2$, then h_t is a $\mathcal{C}^1$ flow.

Foliations with a similar property – namely being locally the product of an interval with a Cantor set – have been studied in [6, 4] in the case of the geodesic flow of homothety surfaces.

The proof of Theorem 1.2 relies on a criterion for establishing a semi-conjugacy between a generalized interval exchange transformation – *GIET* for short – with an interval exchange transformation – *IET* for short – studied by Yoccoz in [27]. This criterion is based on the Rauzy–Veech renormalization algorithm for GIET. An IET is a piece-wise translation bijection, with finitely many branches, of a given base interval, while a GIET is a bijection of the interval which is a piece-wise increasing homeomorphism with finitely many branches. These transformations can be seen as the first return map of a flow on a surface to an interval. See for instance the surveys [16, 29]. On the other hand, IET and GIET can also be seen as generalizations of, respectively, rigid translations and diffeomorphisms of the circle.

Since the work of Denjoy – see [18, 1] – it is known that every $\mathcal{C}^1$ diffeomorphism of the circle such that the logarithm of its derivative is a function of bounded variation has no wandering interval. There is no analogous result concerning interval exchange transformations. In fact, there are several counter-examples, including some very smooth ones. In [22] Levitt found an example of non-uniquely ergodic affine interval exchange transformation – AIET for short – with wandering intervals. Latter, using Rauzy–Veech induction, Camelier and Gutierrez [9] exhibited a uniquely ergodic AIET with wandering intervals, semi-conjugated to a self-similar IET – *i.e.* an IET induced by the foliation of a pseudo-Anosov diffeomorphism. Then Bressaud, Hubert and Maass [7] found a *Galois type* criterion on eigenvalues of a matrix associated to a self-similar IET in order to admit a semi-conjugated AIET with wandering intervals. Finally, Marmi, Moussa and Yoccoz proved [23] that almost every IET admits a semi-conjugated AIET with a wandering interval.

While proving Theorems 1.1 and 1.2, we obtain the following result:

Theorem 1.3. *For all self-similar IET T_0 arising from a pseudo-Anosov map which fixes an Abelian differential, there exists a $\mathcal{C}^1$ GIET T semi-conjugated to T_0 such that*

(i) T has a unique minimal set. This set is a Cantor set and is an attractor for T and T^{-1},
(ii) T is uniquely ergodic,
(iii) T has wandering intervals.

Furthermore T can be chosen to be the Poincaré map of a $\mathcal{C}^1$ flow of a surface and to have a unique wandering interval.

1.1. **Reader's guide.** Section 2 is devoted to the construction of a good candidate for the map f of Theorem 1.1 and to proving the existence of an invariant hyperbolic set constructed as an attractor for f^{-1}. The construction is based on Smale's idea [26, Section I.9] of Derived from Anosov map, and generalizes the construction done by Coudène in [12, Chapter 9] and in [11]. Here, the construction consists in perturbing a given pseudo-Anosov map φ – which, up to taking one of its power, fixes each conical point – by taking benefits of the branched cover surrounding each conical point. From the construction, there exists a natural homotopy between φ and f. To avoid notational complications, we perturb at *all* conical points, but a similar construction can be performed by perturbing only at *some* conical points. All properties hold in that more general setting. We then check basic properties for the map f. In addition, we prove the existence of an invariant compact subset K and show that this set is hyperbolic by computing explicitly a vector field v^s spanning the stable foliation – the unstable foliation coincides with the one of φ.

By construction, v^s satisfies, for all x in K, the relation

$$\mathrm{d}_x f\, v^s(x) = \lambda^{-1} v^s(f(x)) \tag{1.1}$$

where $\lambda > 1$ is the dilation of φ.

Most of the work is carried in Section 3. We prove several technical results concerning the domain and the regularity of v^s. In particular, we prove that v^s extends to $S_g \setminus \Sigma$ into a Lipschitz continuous vector field (Theorem 3.7) which satisfies (1.1) everywhere. Under an additional assumption on the regularity of f, we prove that v^s is $\mathcal{C}^1$ (Theorem 3.8). We also prove that the homotopy between φ and f induces a homotopy between v^s and the constant vector field spanning the stable foliation of φ (Theorem 3.10). By integration of v^s, we get a flow h_t that satisfies

$$f \circ h_t(x) = h_{\lambda^{-1}t} \circ f(x) \tag{1.2}$$

because of (1.1), for all x in $S_g \setminus \Sigma$ and t in $\mathbb{R}$ whenever both sides are well defined. Using this commutation relation we deduce that K is connected and that $f : K \to K$ is topologically transitive. A similar commutation relation is used by Butterley–Simonelli [8] where a parabolic flow is renormalized by a partially hyperbolic map on some 3-dimensional manifold.

In Section 4 we prove that f is mixing with respect to a measure supported by K. Because of the commutation relation (1.2) and the usual functional characterization of mixing, it is sufficient to prove that h_t is uniquely ergodic in order to prove that f is mixing. This last step is done by considering the induced GIET T by h_t to some transversal interval. Using the homotopy of vector fields, we prove that T follows the same full path as a self-similar IET T_0 during the Rauzy–Veech algorithm, hence T is semi-conjugated to T_0 by a result of Yoccoz [27, Proposition 7] – this proves Theorem 1.3. Unique ergodicity of h_t will then follow from writing h_t as the suspension flow of T.

Section 5 is devoted to extending results of Sections 3 and 4 to a slightly different construction than the one in Section 2. There, instead of perturbing a pseudo-Anosov map at conical points, the perturbation is done near a regular fixed point. Most of the results obtained in previous sections hold in this setting.

Finally, in the last section, using extensively Ruelle's results [24] on the SRB measure of Axiom A attractors, we prove that μ is the unique SRB measure of f^{-1} for a $\mathcal{C}^2$ perturbation, and that the correlations decrease exponentially fast for $\mathcal{C}^1$ observables. We also ask whether the result on Ruelle spectrum of linear pseudo-Anosov maps [15] extends to the present case.

2. Perturbations of pseudo-Anosov maps with hyperbolic attractors

In this section we give a method to construct a generalization of Smale's Derived from Anosov maps – which are defined on the two torus – from a perturbation of a given linear pseudo-Anosov map – on a surface of genus $g \geqslant 2$. We then investigate some classical properties of dynamical systems in the case of those specific non-linear pseudo-Anosov transformations. We first prove that these maps are well defined and are homeomorphisms. Then, we show that for some good choices of parameters, conical points are the only attractive fixed points of f. Since the basins of attraction are disjoint open sets and the underlying surface is connected, the complement K of the union of the basins of attraction is not empty. This compact set K is of great importance since in a certain sense the chaotic behaviour of the map f is concentrated in K. More precisely, we show that K is a compact hyperbolic set.

2.1. **Perturbation of a pseudo-Anosov.** Let φ be a *linear* pseudo-Anosov transformation on the Riemann surface S_g of genus g. Therefore the invariant foliations of φ can be derived from a holomorphic quadratic differential q invariant by φ. Up to consider a cover of order two in most cases, it is not too restrictive to assume that the quadratic differential is Abelian, in other words $q = \omega^2$ so that the transition maps, of the flat structure induced by natural coordinates of ω, are pure translations. Up to multiplying ω by a modulus one complex number, the horizontal and vertical foliations $\{\Re(\omega) = 0\}$ and $\{\Im(\omega) = 0\}$ are the invariant foliations of φ. Let $\lambda > 1$ denote the stretch factor of φ. This stretching is assumed to correspond to the vertical measured foliation. The horizontal measured foliation is stretch by a factor λ^{-1}. Let Σ be the set of points where ω vanishes, and we call these points *conical points*. We now consider the flat structure induced by ω on $S_g \smallsetminus \Sigma$, that is charts z so that $\omega = \mathrm{d}z$. In the neighborhood of every conical point $\sigma \in \Sigma$, there exist a positive integer n_σ, an open set and a chart z on this set such that $\omega = z^{n_\sigma - 1}\mathrm{d}z$. The angle around σ is then $2\pi n_\sigma$.

Outside of these neighbourhoods of points of Σ, we set f to be equal to φ. We now construct f to be a perturbation of φ around each σ in Σ.

Let σ be a conical point, V_σ a neighborhood of σ and a chart z on V_σ so that $\omega = z^{n_\sigma - 1}\mathrm{d}z$. Let ξ be the branched cover at σ associated to the chart z, $\xi : z \in z^{-1}V_\sigma \mapsto z^{n_\sigma} \in \xi(z^{-1}(V_\sigma)) \subset \mathbb{C}$. Let $(W_i)_{1 \leqslant i \leqslant 2n_\sigma}$ be a family of open sets of $\mathbb{C} \smallsetminus \mathbb{R}_+$ such that all $\xi|_{W_i}$ are homeomorphisms. Up to replacing φ by one of its power, we assume that every conical point is fixed by φ and that φ respects the leaves of the branched covers: $\varphi(W_i) \cap V_\sigma \subset W_i$ for all i.

We can define f on the base of the branched cover in the exact same manner as Smale [26, Section I.9] does. In order to perform further analysis on the map, we give the following explicit formula that generalized the one used in [12, Chapter 9] and [11] in the case of the cat map on the two-torus.

For $z = x + iy \in \mathbb{C} \smallsetminus \mathbb{R}_+$ in the image of ξ, we define f as :

$$f(\xi|_{W_i}^{-1}(z)) := \xi|_{W_i}^{-1}\left((\lambda + \beta_\sigma k_\sigma(|z|/\alpha_\sigma))x + i\lambda^{-1}y\right),$$

for some $\alpha_\sigma > 0$, $\beta_\sigma < 1 - \lambda$ and with $|z| \leqslant \alpha_\sigma$ and where $k_\sigma : \mathbb{R} \to \mathbb{R}$ is an even unimodal map of class $\mathcal{C}^1$, compactly supported in $[-1, 1]$ and such that k'_σ is Lipschitz continuous, for example $k_\sigma(r) = (1 - r^2)^2 \mathbb{1}_{[-1,1]}$. We do this perturbation at every conical point. We will see that such f is well defined for small enough α_σ.

When such a map f is well defined, we will see in next section that interpolating $(\beta_\sigma)_{\sigma \in \Sigma}$ with 0 gives a homotopy between f and φ. Therefore, f is an example of non-linear pseudo-Anosov transformation.

We give in Figure 1 a heuristic representation, when $n_\sigma = 1$ – which corresponds to the case treated by Smale in [26].

Remark 2.1. Because this construction generalizes Coudène's one [12, Chapter 9] on the two-torus, all results obtained in following sections have their counterparts in the two-torus case.

2.2. **Smoothness and range of parameters.** In order to ensure that the explicit construction introduced above makes sense, we need to ensure that the open sets V_σ near each conical point do not overlap with one another, nor with themselves.

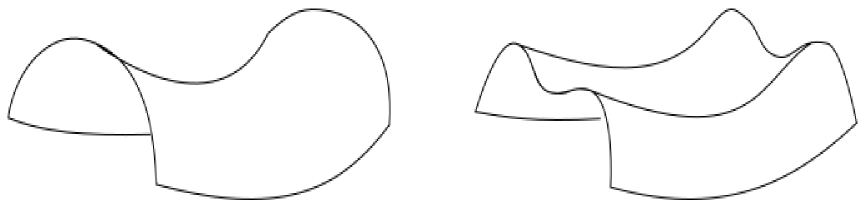


FIGURE 1. Heuristic representations of a saddle and of a perturbed saddle.

This can be easily done by taking the parameter α_σ small enough. We give a simple bound on their size by geometric considerations.

Let $Syst_{s.c}(S_g) = \inf\{\mathrm{d}(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in \Sigma\}$, where d is the distance for the flat metric on S_g associated to the invariant measured-foliation of φ. Let $Syst(S_g) = \inf\{l(\gamma) \mid \gamma \neq 0 \text{ in } \pi_1(S_g)\}$ be the smallest possible length of any non-trivial loop. Then define $\delta_\Sigma = \min(Syst_{s.c}(S_g), Syst(S_g))$.

Proposition 2.2. *For all $\beta_\sigma \in]-\lambda, 0]$ and all $\alpha_\sigma < \delta_\Sigma/2$, f is a homeomorphism on S_g and is a $\mathcal{C}^1$ diffeomorphism on $S_g \smallsetminus \Sigma$.*

Proof. Clearly, f is continuous on S_g and differentiable everywhere except on Σ. The differential on $S_g \smallsetminus \Sigma$ of f is invertible, hence f is a local homeomorphism on $S_g \smallsetminus \Sigma$ and hence $f(S_g \smallsetminus \Sigma)$ is open. In charts around points of Σ, one can see that f is a local homeomorphism in a neighbourhood of Σ. Hence $f(S_g)$ is open. Since S_g is compact, $f(S_g)$ is closed. Hence $f(S_g) = S_g$, because S_g is connected. Therefore, f is a surjective local homeomorphism, hence f is a covering map. Since the pre-image of a point of Σ by f is itself, f is injective. □

By refining the range where the β_σ live, we can turn conical points into attractive fixed points.

Proposition 2.3. *For $\beta_\sigma \in]-\lambda, 1-\lambda[$ and $\alpha_\sigma < \delta_\Sigma/2$, $\sigma \in \Sigma$ is an attractive fixed point for f. Let U_σ be its basin of attraction. Then U_σ is an open set.*

Proof. It is a consequence of the Grobman–Hartman theorem when looking at f through the branched-covering map around σ. We have $U_\sigma = \bigcup_{n\geqslant 0} f^{-n}(B(\sigma, \varepsilon))$, for some small enough $\varepsilon > 0$. □

Since basins of attraction U_σ are disjoint open sets and S_g is connected, these basins are not an open cover. Therefore the complement of the union of basins is not empty. Define $K \coloneqq S_g \smallsetminus \bigsqcup_{\sigma\in\Sigma} U_\sigma$ and $U_\Sigma = \bigsqcup_{\sigma\in\Sigma} U_\sigma$. These sets are clearly invariants by f.

Proposition 2.4. *If for some $\sigma \in \Sigma$, $\beta_\sigma \in]-\lambda, 1-\lambda[$ and $\alpha_\sigma < \delta_\Sigma/2$, then there exists a fixed hyperbolic point p_i^σ, $1 \leqslant i \leqslant 2n_\sigma$, on each vertical ray starting at σ. We number them by going counter-clockwise around σ. All these points are at the same distance $|p^\sigma|$ from σ. Moreover $B(\sigma, |p^\sigma|) \subset U_\sigma$.*

Proof. Let σ, β_σ and α_σ be as in the proposition. Let $\gamma : [0, \alpha_\sigma] \to S_g$ be a unit speed parametrization of a vertical ray such that $\gamma(0) = \sigma$. Hence, in charts, $f(\gamma(t)) = (\lambda + \beta_\sigma k(t/\alpha_\sigma))t$. Let $h : [0, \alpha_\sigma] \to \mathbb{R}$ be the function $h(t) = (\lambda +$

$\beta_\sigma k(t/\alpha_\sigma))t$. Then $h(0)=0$, $h(\alpha_\sigma)=\lambda\alpha_\sigma>\alpha_\sigma$, and $h'(0)=\lambda+\beta_\sigma\in]0,1[$. Hence h has a fixed point in $]0,\alpha_\sigma[$. Call t_0 the smallest fixed point. This value doesn't depend on which vertical ray starting from σ we consider. The point $p=\gamma(t_o)$ is fixed by f and is hyperbolic: in the charts centred at σ, the Jacobian matrix of f at p is

$$(\mathrm{Jac}\, f)(p)=\begin{pmatrix}1+\beta_\sigma t_0\frac{\partial}{\partial x}k(\frac{d(p,\sigma)}{\alpha_\sigma}) & 0\\ 0 & \lambda^{-1}\end{pmatrix},$$

where $\beta_\sigma t_0\frac{\partial}{\partial x}k(\frac{d(p,\sigma)}{\alpha_\sigma})>1$.

By definition of t_0, we have $\gamma([0,t_0[)\subset U_\sigma$. Let $z\in B(\sigma,t_0)$. In the appropriate leaf of the branched-cover over σ, we have $z=(x,y)$ in coordinates. Hence,

$$\begin{aligned}\mathrm{d}(f(z),\sigma)&\leqslant(\lambda+\beta_\sigma k(d(z,\sigma)/\alpha_\sigma))^2x^2+\lambda^{-2}y^2\\ &<(\lambda+\beta_\sigma k(t_0/\alpha_\sigma))^2x^2+\lambda^{-2}y^2\leqslant x^2+\lambda^{-2}y^2\leqslant \mathrm{d}(z,\sigma).\end{aligned}$$

Hence, the function $z\mapsto \mathrm{d}(f(z),\sigma)/\mathrm{d}(z,\sigma)$ is continuous and strictly bounded from above by 1 on the compact annulus $\{z\in S_g\mid\varepsilon\leqslant \mathrm{d}(z,\sigma)\leqslant t_0-\varepsilon\}$. Therefore every orbit of point from the ball $B(\sigma,t_0-\varepsilon)$ ends up entering the ball $B(\sigma,\varepsilon)$. Hence the claim. □

2.3. **Invariant sets.** Here we investigate the topological aspects of the invariant set K. In particular we prove that it can be written as the union of the closure of some stable leaves of the hyperbolic fixed points p_i^σ and that it is a hyperbolic set.

We start by proving that the set U_Σ is dense in S_g, or equivalently that K is of empty interior. In order to do this we need the following lemma which is obtained by simply computing the differential of f.

Lemma 2.5. *Define $(q_i^\sigma)_i$ as the $2n_\sigma$ points at distance $|p^\sigma|$ from σ on the* horizontal *rays starting from σ. Then for all $x\in S_g\smallsetminus\bigsqcup_{\sigma\in\Sigma}(B(\sigma,|p^\sigma|)\cup\{q_i^\sigma\mid 1\leqslant i\leqslant 2n_\sigma\})$, f is a strict dilation in the vertical direction.*

Proposition 2.6. *For all $x\in K$ and all $\varepsilon>0$, every vertical segment of length ε containing x in its interior crosses U_Σ. Hence U_Σ is dense and K has empty interior.*

Proof. By contradiction, let $\gamma:[-\varepsilon,\varepsilon]\to S_g$ be a vertical segment parametrized by arc length, containing some $x\in K$ and such that $\gamma([-\varepsilon,\varepsilon])\cap U_\Sigma=\emptyset$. Without loss of generality, we can assume that $\gamma(0)=x$. Since U_Σ is invariant by f, we see that the existence of some $-\varepsilon\leqslant t\leqslant\varepsilon$ such that $f^n(\gamma(t))\in U_\Sigma$ is impossible. Hence $f^n(\gamma([-\varepsilon,\varepsilon]))\cap U_\Sigma=\emptyset$. By construction of f, the set $f^n(\gamma([-\varepsilon,\varepsilon]))$ is a vertical segment, containing $f^n(x)$ in its interior and of length l_n. Since f is a strict dilation in the vertical direction on the compact set K, there exist $l_*>1$ such that $l_n\geqslant l_*^n$.

Let $\delta=\inf\{|p^\sigma|\mid\sigma\in\Sigma\}$ and since K is compact and invariant by f let $y\in K$ be a subsequential limit of $(f^n(x))_n$. Let n_k be an increasing sequence of integers such that $f^{n_k}(x)$ converges to y as n_k goes to infinity. We know – see [13, corollary 14.15] – that the vertical leaf containing y is at least infinite in one direction and is dense in S_g. In particular, some sufficiently long section of this leaf, containing y, is $\delta/4$-dense in S_g. Hence, for large enough n_k, the curve $f^n(\gamma([-\varepsilon,\varepsilon]))$ is sufficiently long and sufficiently close to the vertical leaf containing y to be $\delta/2$-dense in S_g. In particular, there exists $-\varepsilon<t<\varepsilon$ such that $\mathrm{d}(f^n(\gamma(t)),\sigma)<\delta$ for some $\sigma\in\Sigma$. This contradicts the fact that $B(\sigma,|p^\sigma|)\subset U_\Sigma$. □

Recall definitions of strong stable and strong unstable leaves of $x \in S_g$ with respect to f

$$W^{ss}(x) = \{y \in S_g \mid \mathrm{d}(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\},$$
$$W^{su}(x) = \{y \in S_g \mid \mathrm{d}(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to +\infty\}.$$

If x is a fixed point of f, then these sets are invariant by f.

Here, these leaves at hyperbolic fixed points p_i^σ enable to describe precisely the set K. We start by showing that the stable leaves can be seen as the *accessible border* of U_Σ – and are obviously contained in K. On the other hand, unstable leaves are dense.

Proposition 2.7. *The stable and unstable leaves of the fixed point p_i^σ satisfies the following assertions.*

(i) If $x \in U_\sigma$ and $\gamma : [0,1] \to S_g$ is a vertical curve such that $\gamma(0) = x$, $\gamma([0,1[) \subset U_\sigma$ and $\gamma(1) \notin U_\sigma$ then $\gamma(1)$ belongs to $\bigsqcup_{1 \leqslant i \leqslant 2n_\sigma} W^{ss}(p_i^\sigma)$.

(ii) For all $\sigma \in \Sigma$ and all $1 \leqslant i \leqslant n_\sigma$, the unstable leaf $W^{su}(p_i^\sigma)$ contains a full semi-infinite vertical leaf. Hence $W^{su}(p_i^\sigma)$ is dense in S_g.

Proof. We begin with the first point. Let $\delta > 0$ be the length of the smallest side in the (finite) collection of rectangle neighbourhoods of points p_i^σ given by the Grobman–Hartman theorem. For n large enough, we find that $f^n(x)$ is $\delta/2$-close to some $\sigma \in \Sigma$. Once close to σ by going upward – or downward, depending on the orientation of $f^n \circ \gamma$ – the first time $f^n \circ \gamma$ intersect K is at some point contained in one of the rectangle neighbourhood of some p_i^σ. Therefore this intersection point belongs to $\bigsqcup_{1 \leqslant i \leqslant 2n_\sigma} W^{ss}(p_i^\sigma)$ and is attained at $f^n(\gamma(1))$. The result then follows from the invariance by f of the stable leaves.

We now prove the second point. Let $\gamma : [0, +\infty[\to S_g$ be a unit speed parametrization of the vertical ray starting at $\sigma \in \Sigma$ and containing $p := p_i^\sigma$. In particular, $\gamma(0) = \sigma$ and $\gamma(|p^\sigma|) = p$.

By contradiction, assume there exists $t \geqslant |p^\sigma|$ such that $\gamma(t) \notin W^{su}(p)$. Let $t_0 = \inf\{t \geqslant |p^\sigma| \mid \gamma(t) \notin W^{su}(p)\}$.

We now show that $t_0 > |p^\sigma|$. Let $h : t \mapsto (\lambda + \beta_\sigma k(t/\alpha_\sigma))t$. By construction of f, we have the relation $f(\gamma(t)) = \gamma(h(t))$ for every $t \in [0, \alpha_\sigma[$, and hence $f^n(\gamma(t)) = \gamma(h^n(t))$ for all $n \geqslant 0$. Now $(h^{-1})'(|p^\sigma|) < 1$, so for t close to $|p^\sigma|$, $f^n(\gamma(t)) \to p$ as n goes to infinity. Therefore $t_0 > |p^\sigma|$.

We now prove that $\gamma(t_0)$ is a fixed point of f. We know that $f(\gamma(]|p^\sigma|, t_0[) = \gamma(]|p^\sigma|, s[)$ for some s. But $f(\gamma(]|p^\sigma|, t_0[) \subset W^{su}(p)$. Hence $s \leqslant t_0$.

By contradiction, assume there exists $\varepsilon > 0$ such that $s + \varepsilon < t_0$. So $\gamma([|p^\sigma|, s + \varepsilon[) \subset W^{su}(p)$, and so $f^{-1} \circ \gamma([|p^\sigma|, s + \varepsilon[) \subset W^{su}(p)$. However, $f^{-1} \circ \gamma([|p^\sigma|, s+\varepsilon[) \subset W^{su}(p) = \gamma([|p^\sigma|, t_0 + \delta_\varepsilon[)$ for some $\delta_\varepsilon > 0$ since f is strictly preserving vertical orientation. This contradicts the definition of t_0. Therefore $s = t_0$ and $\gamma(t_0)$ is fixed by f.

The point $\gamma(t_0)$ cannot be in Σ nor be a p_i^σ, otherwise γ would connect two conical points, which is impossible. By computing the differential of f at $\gamma(t_0)$, we see that $\gamma(t_0)$ is a hyperbolic fixed point of f with a vertical unstable leaf. Therefore there exist points whose iterates by f^{-1} converge to p and to $\gamma(t_0) \neq p$. □

These properties of stable and unstable leaves yield to the fact that the set K can be written as a finite union of closure of stable leaves. In fact, we have the following slightly stronger result.

Proposition 2.8. *The compact set K can be written as a finite union of closed invariant sets as follow* $K = \bigcup_{\sigma\in\Sigma}\bigcup_{i=1}^{n_\sigma}\overline{W^{ss}(p_i^\sigma)\cap W^{su}(p_i^\sigma)}$.

Proof. Let $x \in K$ and $\varepsilon > 0$. Let $y \in U_\Sigma$ be in the same vertical leaf as x and obtained by *going downward* by a distance less than ε. Since $U_\Sigma = \bigsqcup_\sigma U_\sigma$, there exists $\sigma \in \Sigma$ such that $y \in U_\sigma$. From the Grobman–Hartman theorem, for each $1 \leqslant i \leqslant 2n_\sigma$ there exists a neighbourhood of p_i^σ on which the dynamic of f is the same as the one of the differential of f. Without loss of generality, we assume that these neighbourhoods are rectangles with vertical and horizontal sides and with centers the p_i^σ's. Up to replacing these rectangles by smaller ones, let δ_σ be a common horizontal size for these rectangles.

For $n \geqslant 0$ large enough, the point y lies in $B(\sigma, \delta_\sigma/4)$. By construction and by the first point of Proposition 2.7, we know that by going *upward* from y we cross some $W^{ss}(p_i^\sigma)$, for some $1 \leqslant i \leqslant 2n_\sigma$. Therefore, by going *upward* from $f^{-n}(y)$ we cross the rectangle of linearisation associated with p_i^σ, and hence the stable leaf $W^{ss}(p_i^\sigma)$ at some point y^u.

Let δ be the modulus of absolute continuity of f^{-n} associated with ε. By density of the unstable leaf of p_i^σ, we can chose a point z such that $\mathrm{d}(f^n(z), f^n(y)) < \min(\delta, \delta_\sigma/4)$ so that by going *upward* from $f^n(z)$ we cross $W^{ss}(p_i^\sigma)$ at some point z^u, at distance less than δ from y^u. Finally, the point $f^{-n}(z^u) \in W^{ss}(p_i^\sigma)\cap W^{su}(p_i^\sigma)$ is at distance less than 3ε from x. □

Finally, we explicit stable and unstable foliations such that the set K is hyperbolic with respect to f. To do this, we compute a vector field that is uniformly contracted by the differential of f.

Theorem 2.9. *The set K is hyperbolic. The invariant distributions are $E^u(x) = \mathbb{R}e_v$ and $E^s(x) = \mathbb{R}v^s(x)$, where*

$$v^s(x) \coloneqq e_h - \sum_{i\geqslant 0}\lambda^{-i} b(f^i(x))\prod_{j=0}^{i}\frac{1}{a(f^j(x))}\, e_v,$$

with $a(x) \coloneqq \langle \mathrm{d}_x f \cdot e_h, e_h\rangle$ and $b(x) \coloneqq \langle \mathrm{d}_x f \cdot e_v, e_h\rangle$. In particular, v^s satisfies $\mathrm{d}f\, v^s = \lambda^{-1} v^s \circ f$ on K.

Proof. We will explicit the stable and the unstable directions of the splitting of the tangent space. Write the differential of f at $x \in S_g \smallsetminus \Sigma$ in the basis (e_h, e_v)

$$\mathrm{d}_x f = \begin{pmatrix} a(x) & b(x) \\ 0 & \lambda^{-1}\end{pmatrix}.$$

Therefore, for every positive integer n, we have the following,

$$\begin{aligned}\mathrm{d}_x(f^n) &= \mathrm{d}_{f^{n-1}(x)} f \cdots \mathrm{d}_{f(x)} f\, \mathrm{d}_x f,\\ &= \begin{pmatrix} a(f^{n-1}(x)) & b(f^{n-1}(x)) \\ 0 & \lambda^{-1}\end{pmatrix}\cdots\begin{pmatrix} a(f(x)) & b(f(x)) \\ 0 & \lambda^{-1}\end{pmatrix}\begin{pmatrix} a(x) & b(x) \\ 0 & \lambda^{-1}\end{pmatrix},\\ &= \begin{pmatrix} A_n(x) & B_n(x) \\ 0 & \lambda^{-n}\end{pmatrix}.\end{aligned}$$

We have that $A_n(x) = \prod_{i=0}^{n-1} a(f^i(x))$. We compute B_n explicitly. This sequence satisfies a recursive formula, which can be solved

$$\begin{aligned} B_{n+1} - a(f^n)B_n &= \lambda^{-n} b(f^n), \\ B_{n+1}/A_{n+1} - B_n/A_n &= \lambda^{-n} b(f^n)/A_{n+1}, \\ B_n/A_n &= \sum_{i=0}^{n-1} \lambda^{-i} b(f^i)/A_{i+1}. \end{aligned}$$

Finally we get

$$\frac{B_n}{A_n}(x) = \sum_{i=0}^{n-1} \lambda^{-i} b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))}.$$

We can now explicit the eigenvectors of $\mathrm{d}_x(f^n)$. The obvious one, associated with the eigenvalue $A_n(x)$, is e_v. The other one is

$$v_n(x) = \begin{pmatrix} -B_n(x)/A_n(x) \\ 1 - \lambda^{-n}/A_n(x) \end{pmatrix} = \begin{pmatrix} -\sum_{i=0}^{n-1} \lambda^{-i} b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))} \\ 1 - \prod_{i=0}^{n-1} \frac{1}{\lambda a(f^i(x))} \end{pmatrix}.$$

We now study the convergence of the v_n's as n goes to infinity. First, since $a > 1$, b are continuous functions over the compact set K, there exist constants a^* and C such that $a > a^* > 1$ and $|b| < C$. Therefore, the second coordinate converges to 1 as n goes to infinity. For the first coordinate, we have the uniform bound over K

$$\sum_{i=0}^{n-1} \left| \lambda^{-i} b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))} \right| \leqslant C \sum_{i=0}^{n-1} (\lambda a^*)^{-i} \leqslant C \frac{\lambda a^*}{\lambda a^* - 1}.$$

Hence, the series of continuous functions converges uniformly over K to a continuous function. Call v^s the limit of v_n as n goes to infinity.

A short computation shows that for all x in K, v^s satisfies $\mathrm{d}_x f \cdot v^s(x) = \lambda^{-1} v^s(f(x))$. Finally, we get the following splitting of the tangent space at each x in K, $T_x S_g = \mathbb{R} v^s(x) \oplus \mathbb{R} e_v$, so that K is a hyperbolic set. □

3. Smoothness of the stable foliation and renormalized flow

In this section we prove that v^s can be extended to the whole set $S_g \setminus \Sigma$ of regular points, such that the extension is still uniformly contracted by the action of f and so that it is Lipschitz continuous. Under further assumption on the smoothness of f, we prove that v^s is $\mathcal{C}^1$. Furthermore, in view of the next section, we prove that v^s depends continuously on the parameter β – occuring in the construction of f. This regularity property is a central point in the proof of the unique ergodicity of the flow, which implies the mixing of f with respect to the very same invariant measure thanks to the commutation relation between f and h_t. Since v^s is Lipschitz continuous, it can be integrated into a continuous flow h_t which enjoys a nice commutation relation with f – in other words, f renormalizes h_t. From the properties of h_t, we show that the set K is connected, transverse to any vertical leaf, and that f is transitive with respect to the trace topology on K.

3.1. **Construction of a useful open cover of S_g.** In order to proceed, we first need to construct an open cover of $S_g \smallsetminus \Sigma$ such that f satisfies some nice estimates on elements of this cover. This is done in the following proposition.

Proposition 3.1. *For all $\varepsilon > 0$ small enough, there exist $\eta > 0$, $\delta > 0$, and an open cover $S_g = A_\eta \cup \bigsqcup_{\sigma\in\Sigma} B_{\sigma,\delta}$ such that $a > 1+\eta$ on A_η and $\mathrm{d}(f(x),\sigma) < (1-\delta)\mathrm{d}(x,\sigma)$ on $B_{\sigma,\delta} \smallsetminus \{\sigma\}$.*

Proof. By continuity of f, there exists an $\varepsilon > 0$ such that

$$\{x \in V_\sigma \mid \mathrm{d}(f(x),\sigma) < \mathrm{d}(x,\sigma)\} \supset B(\sigma, |p^\sigma|) \cup \bigcup_{i=1}^{2n_\sigma} B(q_i^\sigma, \varepsilon) =: B_\sigma^\varepsilon,$$

for all σ, where $V_\Sigma = \bigsqcup_{\sigma\in\Sigma} V_\sigma$ is the open neighbourhood of Σ on which $f \not\equiv \varphi$.

Since $S_g \smallsetminus \bigsqcup_{\sigma\in\Sigma} B_\sigma^\varepsilon$ is compact and $a > 1$ on it, there exists $\eta > 0$ such that $a > 1 + 2\eta$ on this compact set. Call $A_\eta = \{x \in S_g \mid a > 1+\eta\}$. By construction, $S_g = A_\eta \cup \bigcup_{\sigma\in\Sigma} B_\sigma^\varepsilon$.

Since all B_σ^ε are open sets, radial and centred on σ, we have $B_\sigma^\varepsilon = \bigcup_{n\geqslant 1} \left(1 - \frac{1}{n}\right) B_\sigma^\varepsilon$. Now, by compactness of S_g, there exists n_0 such that:

$$S_g = A_\eta \cup \bigcup_{\sigma\in\Sigma} \left(1 - \tfrac{1}{n_0}\right) B_\sigma^\varepsilon.$$

On a small open neighbourhood W_σ of σ, by construction of f we have that $\mathrm{d}(f(x),\sigma)/\mathrm{d}(x,\sigma) < C < 1$. Now, on the compact set $\overline{(1-\frac{1}{2n_0})B_\sigma^\varepsilon} \smallsetminus W_\sigma$, the continuous function $\mathrm{d}(f(x),\sigma)/\mathrm{d}(x,\sigma)$ is positive and strictly bounded from above by 1. On the other hand, up to shrinking W_σ, the function $\mathrm{d}(f(x),\sigma)/\mathrm{d}(x,\sigma)$ is bounded on $W_\sigma \smallsetminus \{\sigma\}$ by $\max(\lambda^{-1}, \lambda + \beta_\sigma + \tilde{\delta}) < 1$, for some small $\tilde{\delta} > 0$. Hence, there exists $\delta > 0$, independent of σ, such that for all x in $(1-\frac{1}{n_0})B_\sigma^\varepsilon \smallsetminus \{\sigma\}$, $\mathrm{d}(f(x),\sigma) < (1-\delta)\mathrm{d}(x,\sigma)$. We then call $B_{\sigma,\delta} = (1-\frac{1}{n_0})B_\sigma^\varepsilon$. ☐

3.2. **Lipschitz extension of v^s to $S_g \smallsetminus \Sigma$.** Here we prove that the infinite sum in the definition of the vector field v^s on K does converge on all $S_g \smallsetminus \Sigma$. This way we can define v^s on $S_g \smallsetminus \Sigma$. Furthermore, we prove that this extended vector field is Lipschitz continuous.

We proceed in two steps. First we show that v^s is bounded and continuous on $S_g \smallsetminus \Sigma$. To do this, we need several lemmas which follow directly from computation of $\mathrm{d}f$.

Lemma 3.2. *The partial derivative $b = \langle \mathrm{d}f(e_v), e_h\rangle$ of f is locally Lipschitz in some neighbourhood of Σ. Furthermore, by continuity we can set $b(\sigma) = 0$ for each $\sigma \in \Sigma$.*

Lemma 3.3. *On each basin U_σ, the partial derivative $a = \langle \mathrm{d}f(e_h), e_h\rangle$ of f is bounded from below by $\lambda + \beta_\sigma$.*

Theorem 3.4. *If $\beta_\sigma \in]-\lambda+\lambda^{-2}, -\lambda+1[$ for all σ in Σ, then the vector field v^s is bounded and continuous on $S_g \smallsetminus \Sigma$. Furthermore, by construction, the formula $\mathrm{d}f(v^s) = \lambda^{-1} v^s \circ f$ holds on $S_g \smallsetminus \Sigma$.*

Proof. Call $s_i = \lambda^{-i} b \circ f^i \prod_{j=1}^{i} \frac{1}{a \circ f^j}$. Let V be a neighbourhood of some σ such that b is Lipschitz on it and f contracts by a factor $\max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma) < 1$. Without loss of generality, we assume that V is a ball centred at σ of radius ε and that $f(V) \subset V$. Since $U_\sigma = \bigcup_{N \geqslant 0} f^{-N} V$, for all $x \in U_\sigma$ there exist some $N = N(x)$ and an integer n_V which only depends on V, such that for all $n \geqslant N$, $f^n(x) \in V$, at most n_V points of the orbits fall into $B_{\sigma,\delta} \setminus V$ and the rest lives in A_η.

Let $x \in U_\sigma$, $x \neq \sigma$. Since $U_\sigma = \bigcup_{n \geqslant 0} f^{-n} V$, let N be the smallest integer such that $f^N(x) \in V$. We distinguish three cases :

- $i \leqslant N - n_V$. Therefore $|s_i(x)| \leqslant \lambda^{-i} \left(\frac{1}{1+\eta}\right)^{i+1} \sup |b|$.
- $N - n_V < i \leqslant N$. Hence $|s_i(x)| \leqslant \lambda^{-i} \left(\frac{1}{1+\eta}\right)^{N-n_V} \left(\frac{1}{\lambda+\beta}\right)^{i-(N-n_V)} \sup |b|$.
- $i = j + N > N$. We get $|s_i(x)| \leqslant \lambda^{-(j+N)} \left(\frac{1}{\lambda+\beta}\right)^{j+N} \mathrm{Lip}(b) \varepsilon \max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma)^j$.

Therefore, if $\lambda^{-2} < \lambda + \beta_\sigma$, then

$$\sum_{i \geqslant 0} |s_i(x)| \leqslant \sup |b| \frac{\lambda(1+\eta)}{\lambda(1+\eta) - 1} \left(1 + \sum_{i=0}^{n_V} \left(\frac{1}{\lambda + \beta}\right)^i\right) + \frac{\mathrm{Lip}(b)\varepsilon}{1 - \frac{\max(\lambda^{-1}, \lambda+\beta_\sigma+\delta_\sigma)}{\lambda(\lambda+\beta_\sigma)}},$$

which is uniform in x on U_σ. Hence, the convergence is uniform on the compact subsets of $U_\sigma \setminus \{\sigma\}$ and $\sum s_i$ is continuous on $U_\sigma \setminus \{\sigma\}$, for all $\sigma \in \Sigma$.

We now show that this function defined on $U_\Sigma = \sqcup U_\sigma$ can be extended by continuity on K. Call $u(x) = e_h - \sum_{i \geqslant 0} s_i(x) e_v$ the vector based at $x \in S_g \setminus \Sigma$.

Let $x \in K$ and, by density of U in S_g, $(x_n)_n \in U^{\mathbb{N}}$ such that $x_n \to x$ as n goes to infinity. Since $(u(x_n))_n$ is bounded, up to extracting, the sequence converges to some u_0. Furthermore, by a diagonal argument and up to extracting, $u(f^k(x_n)) \to u_k$ for all $k \in \mathbb{Z}$ as n goes to infinity. Now, by construction of u, $\mathrm{d}f(u) = \lambda^{-1} u \circ f$. Hence, by continuity of f and $\mathrm{d}f$, $\mathrm{d}_x(f^k)(u_0) = \lambda^{-k} u_k$. We now show that $u_0 = v^s(x)$. By hyperbolicity of K, there exist real numbers x_s, x_u such that $u_0 = x_s v^s(x) + x_u e_v$. Therefore, by hyperbolicity of K,

$$\begin{aligned} |x_u| &= ||x_u e_v||, \\ &= ||\mathrm{d}_{f^k(x)} f^{-k} \mathrm{d}_x f^k x_u e_v||, \\ &\leqslant C(\tfrac{1}{a_*})^k ||\mathrm{d}_x f^k x_u e_v||, \\ &= C(\tfrac{1}{a_*})^k ||\mathrm{d}_x f^k (u_0 - x_s v^s(x))||, \\ &\leqslant C(\tfrac{1}{a_*})^k \lambda^{-k} (\sup ||u|| + x_s \sup ||v^s||), \end{aligned}$$

which goes to zero as k goes to infinity. Hence $u_0 = x_s v^s(x)$. Now both u_0 and $v^s(x)$ have the same non-zero coordinate along e_v in the base (e_v, e_h). Hence $u_0 = v^s(x)$. Finally, u extends continuously on K by v^s. We call v^s this vector field on $S_g \setminus \Sigma$. □

We can now present the proof of the Lipschitz continuity of v^s on $S_g \setminus \Sigma$. To this end, we need a few more estimates on the differential of f and on its coefficients.

Lemma 3.5. *For all $x \in S_g \setminus \Sigma$, we have the following estimate $\frac{\|\,\mathrm{d}_x f^n\|}{A_n(x)} \leqslant 2\max\left(1, \frac{|B_n|(x)+\lambda^{-n}}{A_n(x)}\right)$. In particular, $\|\,\mathrm{d}f^n\|/A_n$ is bounded on $\bigcup_{i=0}^{n} f^{-i}A_\eta$. Furthermore, the bound B can be chosen independently of n.*

Proof. By a direct computation, for $(u,v) \coloneqq ue_h + ve_v$

$$\begin{aligned}\|\,\mathrm{d}_x f^n(u,v)\|^2 &= (A_n(x)u + B_n(x)v)^2 + (\lambda^{-1}v)^2,\\ &\leqslant 4A_n(x)^2u^2 + (4B_n(x)^2 + \lambda^{-2n})v^2,\\ &\leqslant 4\max(A_n(x)^2, B_n(x)^2 + \lambda^{-2n})\|(u,v)\|^2.\end{aligned}$$

For $x \in \bigcup_{i=0}^{n} f^{-i}A_\eta$, we know that $\lambda^{-k}/A_k(x) < (\lambda(1+\eta))^{-k}$ and that $-B_n/A_n$ is the partial sum of $\sum s_i$, hence uniformly bounded. □

The following lemma is a direct consequence of the Lipschitz continuity of k' intervening in the construction of f.

Lemma 3.6. *The functions a and $\frac{1}{a}$ are Lipschitz continuous on S_g.*

Theorem 3.7. *If $\beta_\sigma \in]-\lambda+\lambda^{-2}, -\lambda+1[$ for all σ in Σ, then the vector field v^s is Lipschitz continuous on $S_g \setminus \Sigma$.*

Proof. Since all of the partial sums of $\sum s_i$ are Lipschitz continuous, we give summable estimates of local Lipschitz constants. Let $x \in U_\sigma$. Let V, $N = N(x)$ and n_V be as in the proof of Theorem 3.4. Therefore $U_\sigma = \bigcup_{n\geqslant 0} f^{-n}V$. We use the notation $\mathrm{Lip}_x(g)$ to indicate the local Lipschitz constant of a function g in at least one neighbourhood of x.

Let $\varepsilon > 0$. On a small enough neighbourhood of x, we have that $\mathrm{Lip}_x(f^j) \leqslant (1+\varepsilon)\|\,\mathrm{d}_x f^j\|$ and $\sup \frac{1}{A_j} \leqslant (1+\varepsilon)\frac{1}{A_j(x)}$ for all $j \leqslant i$. We distinguish the three following cases:

- $i \leqslant N - n_V$. We have directly that,

$$\begin{aligned}\mathrm{Lip}_x(s_i) &\leqslant \lambda^{-i}\left(\mathrm{Lip}(b)\ \mathrm{Lip}(f^i)\sup\tfrac{1}{A_i} + \sup(b\circ f^i)\ \mathrm{Lip}\tfrac{1}{a}\sum_{j=0}^{i}\mathrm{Lip}(f^j)\sup\tfrac{1}{A_{j-1}}\sup\tfrac{A_j}{A_i}\right),\\ &\leqslant \lambda^{-i}B(1+\varepsilon)^2\left(\mathrm{Lip}(b) + \sup|b|\ \mathrm{Lip}\tfrac{1}{a}\sup(a)\sum_{j=0}^{i}\left(\frac{1}{1+\eta}\right)^j\right),\\ &\leqslant C_{\perp\!\!\!\perp i,N,x}\,\lambda^{-i},\end{aligned}$$

where $C_{\perp\!\!\!\perp i,N,x}$ stands for a constant independent of i, N and x.

- $N - n_V \leqslant i < N$. Up to multiplying some part of the above estimate by $(\frac{1}{\lambda+\beta_\sigma})^{n_V}$, we have:

$$\mathrm{Lip}_x(s_i) \leqslant C_{\perp\!\!\!\perp i,N,x}\,\lambda^{-i}.$$

- $i = l + N \geqslant N$. In this case, the following estimates hold:

$$\begin{aligned}
\mathrm{Lip}_x(b\circ f^{l+N}) \sup \frac{1}{A_{l+N}} &\leqslant \mathrm{Lip}(b)\ \mathrm{Lip}(f^N) \sup \frac{1}{A_N}\ \mathrm{Lip}_{f^N(x)}(f^l) \sup \frac{A_N}{A_{l+N}},\\
&\leqslant Lip(b)(1+\varepsilon)^2 \frac{\|\,\mathrm{d}_x f^N\|}{A_N(x)} \max(\lambda^{-1}, \lambda+\beta_\sigma+\delta_\sigma)^l \left(\frac{1}{\lambda+\beta_\sigma}\right)^l,\\
&\leqslant C_{\perp\!\!\perp x,i,N}\ \max\left(\frac{\lambda^{-1}}{\lambda+\beta_\sigma}, 1+\frac{\delta_\sigma}{\lambda+\beta_\sigma}\right)^l.
\end{aligned}$$

$$\begin{aligned}
\sup(b\circ f^{l+N})\ \mathrm{Lip}_x \frac{1}{A_{l+N}} &\leqslant \varepsilon \max(\lambda^{-1}, \lambda+\beta_\sigma+\delta_\sigma)^l\ \mathrm{Lip}\frac{1}{a} \left(\sum_{j=0}^{N-1} Lip(f^j) \sup\frac{1}{A_{j-1}} \sup \frac{A_j}{A_{l+N}}\right.\\
&\qquad \left. + \sum_{j=0}^{l} Lip(f^N)\ \mathrm{Lip}_{f^N(x)}(f^l) \sup \frac{1}{A_N} \sup(a) \sup \frac{A_N}{A_{l+N}}\right),\\
&\leqslant \varepsilon \max(\lambda^{-1}, \lambda+\beta_\sigma+\delta_\sigma)^l\ \mathrm{Lip}\frac{1}{a} C_{\perp\!\!\perp x,i} \left(\frac{1}{\eta} + n_V \left(\frac{1}{\lambda+\beta_\sigma}\right)^{n_V}\right.\\
&\qquad \left. + \sup(a) \sum_{j=0}^{l} \max\left(\frac{\lambda^{-1}}{\lambda+\beta_\sigma}, 1+\frac{\delta_\sigma}{\lambda+\beta_\sigma}\right)^j\right).
\end{aligned}$$

These two bounds are independent of N, hence of x.

By setting $\beta_\sigma \in]-\lambda+\lambda^{-2}, -\lambda-1[$, all the bounds on $\mathrm{Lip}_x(s_i)$ decay geometrically. Hence all partial sums of $\sum s_i$ share a common Lipschitz constant near each point of U, independent of the base-point.

We give now some estimates when $x \in K$. Therefore $f^n(x) \in A_\eta$ for all n. The following estimate holds:

$$\begin{aligned}
\mathrm{Lip}_x(s_i) &\leqslant \lambda^{-i} \left(\mathrm{Lip}(b)\ \mathrm{Lip}(f^i) \sup \frac{1}{A_i} + \sup|b|\ \mathrm{Lip}\left(\frac{1}{a}\right) \sum_{j=0}^{i} \mathrm{Lip}(f^j) \sup \frac{1}{A_{j-1}} \sup \frac{A_j}{A_i}\right),\\
&\leqslant \lambda^{-i}(1+\varepsilon)^2 B \left(\mathrm{Lip}(b) + \sup|b| \sup(a)\ \mathrm{Lip}\left(\frac{1}{a}\right) \sum_{j=0}^{i} \left(\frac{1}{1+\eta}\right)^j\right),\\
&\leqslant C_{\perp\!\!\perp x,i}\, \lambda^{-i}.
\end{aligned}$$

Finally, every partial some of $\sum s_i$ shares a common Lipschitz constant on $S_g \smallsetminus \Sigma$. Therefore v^s is Lipschitz continuous on $S_g \smallsetminus \Sigma$. □

3.3. **Differentiability of v^s.** Here we prove that when the function k is $\mathcal{C}^2$, the stable vector field v^s is $\mathcal{C}^1$. In order to prove this result, we use similar computations as in the proof of Theorem 3.7 and show that v^s is differentiable on every compact set of U and on K. We then use the relation $\mathrm{d}f\, v^s = \lambda^{-1} v^s \circ f$ (more precisely, the differential of this relation) in order to prove that there is a unique extension of $\mathrm{d}v^s$ from U to $S_g \smallsetminus \Sigma$, and it coincides with $\mathrm{d}v^s$ on K.

Theorem 3.8. *If the function $k : \mathbb{R} \to \mathbb{R}$ in the construction of f is also $\mathcal{C}^2$, then the vector field v^s is C^1.*

Corollary 3.9. *If k is $\mathcal{C}^2$, then h_t is a $\mathcal{C}^1$ flow.*

Proof. From the same estimates as in the proof of Theorem 3.7, we get that the series of differentials $\sum_{i\geqslant 0} \mathrm{d}s_i$ converges uniformly on K and on compact subsets of $U\setminus\Sigma$. By uniform converge, v^s is therefore differentiable on K and on $U\setminus\Sigma$, but we still need to prove that $x\mapsto \mathrm{d}_x v^s$ is continuous on $S_g\setminus\Sigma$. To this end, we use the fact that v^s is uniformly contracted by f.

By design, v^s satisfies the equality $\mathrm{d}_x f\, v^s(x)=\lambda^{-1}v^s(f(x))$ for all $x\notin\Sigma$. Now, by differentiation, we get for all x in U

$$\mathrm{d}_x^2 f(v^s(x),\cdot)+\mathrm{d}_x f\,\mathrm{d}_x v^s=\lambda^{-1}\mathrm{d}_{f(x)}v^s\,\mathrm{d}_x f. \tag{3.1}$$

Let $x\in K$ and $(x_n)_n$ be a sequence in U converging to x as n goes to infinity. By the Arzelà–Ascoli theorem, in order to prove that $(\mathrm{d}_{x_n}v^s)_n$ converges to $\mathrm{d}_x v^s$, it is sufficient to prove that $(\mathrm{d}_{x_n}v^s)_n$ has a unique subsequential limit.

To be exact, in order to apply the Arzelà–Ascoli theorem, we need the maps to have a compact domain. We address this problem by associating to any linear map $l:\mathbb{R}^d\to\mathbb{R}^d$ its restriction to the unit sphere $\tilde{l}:\mathbb{S}^{d-1}\to\mathbb{R}^d$, in addition with the closed condition

$$||x+\theta y||\,\tilde{l}\left(\frac{x+\theta y}{||x+\theta y||}\right)=\tilde{l}(x)+\theta\tilde{l}(y),\qquad x,y\in\mathbb{S}^{d-1},\,\theta\in\mathbb{R}. \tag{3.2}$$

Now, any linear map can be built from a map on the sphere satisfying the condition (3.2). This one-to-one correspondence is enough to overcome the issue of non-compactness of the domain.

Let u_x be a subsequential limit of $(\mathrm{d}_{x_n}v^s)_n$. Using (3.1) and the fact that f is $\mathcal{C}^2$, we also get a subsequential limit $u_{f(x)}$ of $(\mathrm{d}_{f(x_n)}v^s)_n$. By the same process, we get for all integer k a subsequential limit $u_{f^k(x)}$ of $(\mathrm{d}_{f^k(x_n)}v^s)_n$ so that

$$\mathrm{d}^2_{f^k(x)}f(v^s(f^k(x)),\cdot)+\mathrm{d}_{f^k(x)}f\,u_{f^k(x)}=\lambda^{-1}u_{f^{k+1}(x)}\,\mathrm{d}_{f^k(x)}f.$$

Taking the difference with (3.1) we get, after induction, that for all integer k

$$\mathrm{d}_x f^k(\mathrm{d}_x v^s-u_x)=\lambda^{-k}(\mathrm{d}_{f^k(x)}-u_{f^k(x)})\mathrm{d}_x f^k \tag{3.3}$$

We now prove that the difference $\alpha_0\coloneqq \mathrm{d}_x v^s-u$ is the zero map. First, notice that since $v^s(x)=e_h-(\Sigma_i s_i(x))e_v$, we must have $\mathrm{Im}(\mathrm{d}_x v^s)\subset\mathbb{R}e_v=E^u(x)$ and, by taking limits, $\mathrm{Im}(u_x)\subset E^u(x)$. Therefore $\mathrm{Im}(\alpha_0)\subset E^u(x)$. Since v^s is Lipschitz continuous, the operators $\mathrm{d}_{f^k(x)}v^s-u_{f^k(x)}$ are uniformly bounded. Therefore, from the hyperbolicity of K and the relation (3.3), we get that $\alpha_0(\mathbb{R}v^s(x))\subset\mathbb{R}v^s(x)$, and so $\alpha_0(v^s(x))=0$. Since $(v^s(x),e_v)$ is a basis of $\mathbb{R}^2$, there exists some real number α such that $\alpha(e_v)=\alpha e_v$. Applying (3.3) to e_v, we get that

$$0=(\alpha\mathrm{Id}-\lambda^{-k}(\mathrm{d}_{f^k(x)}f-u_{f^k(x)}))\mathrm{d}_x f\,e_v.$$

If α is not zero, then for large enough value of k the map $(\mathrm{Id}-\frac{\lambda^{-k}}{\alpha}(\mathrm{d}_{f^k(x)}f-u_{f^k(x)}))\mathrm{d}_x f$ is invertible, hence a contradiction. Therefore $\alpha=0$ and $u=\mathrm{d}_x v^s$. Finally, we get that $x\in U\setminus\Sigma\mapsto\mathrm{d}_x v^s$ extends continuously, in a unique fashion, to $S_g\setminus\Sigma$. □

3.4. **Continuity of v^s with respect to β.** In the next section we prove that h_t is uniquely ergodic and that f is mixing with respect to the invariant measure of h_t. To do so, we first prove that the family of vector fields v^s is smooth with respect to the amplitude parameter β in the definition of f.

We will use the following notations. For all $\beta = (\beta_\sigma)_{\sigma\in\Sigma}$, write f_β the function f with the amplitude parameter β, and v^s_β its corresponding vector field. We also assume the parameter $(\alpha_\sigma)_{\sigma\in\Sigma}$ to be fixed.

In this section, we only consider the case $\#\Sigma = 1$, hence the vector β has only one component. The general case leads to very similar computations.

More precisely, we prove the following theorem.

Theorem 3.10. *The map $\beta \in]-\lambda+\lambda^2, 0] \mapsto v^s_\beta$ is continuous for the sup-norm. As a consequence, the function $(x,\beta) \mapsto v^s_\beta(x)$ is continuous on $(S_g \setminus \Sigma) \times]-\lambda+\lambda^{-2}, 0]$.*

To show this continuity, we split the domain into three subsets. We will need the following lemma.

Lemma 3.11. *For all β in $]-\lambda+\lambda^{-2}, 0]$, the eigenspace of $(f_\beta)_* := (\mathrm{d}f_\beta)^{-1} U_{f_\beta}$ associated with the eigenvalue λ is of dimension one when acting on the space of bounded and continuous vector fields on the tangent vector bundle of $S_g \setminus \Sigma$, where U_f stands for the Koopman operator of f.*

Proof. Let $\beta \in]-\lambda+\lambda^{-2}, 0]$. Let w be a vector field in the eigenspace of $(f_\beta)_*$ associated with the eigenvalue λ. In other words, w is such that $\mathrm{d}_x f_\beta(w(x)) = \lambda^{-1} w(f_\beta(x))$, for all x. Now, since v^s is continuous, non vanishing and transverse to e_v, there exist two functions w_1 and w_2 uniquely determined such that $w(x) = w_1(x)v^s(x) + w_2(x)e_v$ for all x. These two functions are bounded and continuous. Hence, we have,

$$\begin{aligned} \mathrm{d}_x f_\beta(w_2(x)e_v) = a(x)w_2(x)e_v &= \mathrm{d}_x f_\beta(w(x) - w_1(x)v^s(x)), \\ &= \lambda^{-1}(w(f_\beta(x)) - w_1(x)v^s(f_\beta(x))), \\ w(f_\beta(x)) &= w_1(x)v^s(f_\beta(x)) + \lambda a(x) w_2(x) e_v. \end{aligned}$$

Therefore, w_1 is invariant by f_β and for all $i > 0$,

$$w_2(x) = \prod_{j=0}^{i-1} \frac{1}{\lambda a(f^j_\beta(x))} w_2(f^i_\beta(x)).$$

By continuity of w_2 and compactness of S_g, w_2 is bounded. Now, we distinguish two cases in order to prove that $w_2 = 0$.

For $\beta_\sigma < 1-\lambda$, there exists a fixed point p^σ_i, in K, whose unstable leaf is dense. Since at this point $a(p^\sigma_i) > 1$, by continuity of a, we get that $a > 1$ in a neighbourhood of p^σ_i, hence $1/(\lambda a) < \lambda^{-1} < 1$ and $w_2 = 0$.

For $1-\lambda \leqslant \beta_\sigma \leqslant 0$, we know that the unstable leaf of σ is dense in S_g. By continuity on every leaf of the branched cover at σ, we can set $a(\sigma) = \lambda + \beta_\sigma \geqslant 1$. Hence, in a neighbourhood of σ, we get $1/(\lambda a) \leqslant \lambda^{-1} < 1$, hence $w_2 = 0$.

In order to prove that w_1 is constant, we also distinguish two cases.

For $\beta_\sigma < 1-\lambda$, the unstable leaf of each p^σ_i is dense. Hence $w_1(x) = w_1(p^\sigma_i)$ for all x. Hence the claim in this case.

For $1-\lambda \leqslant \beta_\sigma \leqslant 0$, the unstable leaf of σ is dense. Therefore, $w_1(x) = w_1(\sigma)$ for all x. Hence the claim. □

Proof of Theorem 3.10. We first prove that $||v^s_\beta - v^s_{\beta_0}||_\infty \xrightarrow[\beta\to\beta_0]{} 0$ for all β_0 in $]-\lambda+\lambda^{-2}, 1-\lambda[$. From proofs of Theorems 3.4 and 3.7, we can see that on a small enough neighbouhood B_0 of β_0, the vector fields v^s_β are uniformly bounded, as well as their Lipschitz constants. By the Arzelà-Ascoli theorem, the set $\{v^s_\beta \mid \beta \in B_0\}$ is relatively compact. Take a sequence of $(\beta_n)_n$ converging to β_0, then every subsequential limit w of $(v^s_{\beta_n})_n$ must satisfies $(f_{\beta_0})_* w = \lambda w$. By Lemma 3.11, the space of such vector fields is one dimensional, hence there exists a constant c such that $w = c v^s_{\beta_0}$. Since in the basis (e_h, e_v) all the component of v^s_β along e_h is 1, we get that $c = 1$. Hence $v^s_{\beta_n}$ converges uniformly to $v^s_{\beta_0}$, and so for all sequences $(\beta_n)_n$. The rest of the claim follows directly by the triangle inequality and Lipschitz continuity.

We now prove that $||v^s_\beta - v^s_{\beta_0}||_\infty \xrightarrow[\beta \to \beta_0]{} 0$ for all $\beta_0 \in [1-\lambda, 0]$. The same argument as in the case above holds. Indeed, for all $\beta \in [1-\lambda, 0]$ we get

$$\sum_{i \geqslant 0} \left| \lambda^{-i} b_\beta \circ f^i_\beta \prod_{j=0}^{i} \frac{1}{a_\beta \circ f^j_\beta} \right| \leqslant \frac{1}{1-\lambda^{-1}} ||b_\beta||_\infty.$$

Hence v^s_β is uniformly bounded for β in a neighbourhood of β_0. Similarly, the following estimate on the Lipschitz constant holds for all $\varepsilon > 0$

$$\sum_{i \geqslant 0} \mathrm{Lip}_x \left(\lambda^{-i} b_\beta \circ f^i_\beta \prod_{j=0}^{i} \frac{1}{a_\beta \circ f^j_\beta} \right) \leqslant (1+\varepsilon)^2 ||v^s_\beta||_\infty \sum_{i \geqslant 0} \lambda^{-i} \Bigg(\mathrm{Lip}(b_\beta) + i \, \mathrm{Lip}\left(\frac{1}{a_\beta}\right) ||a_\beta||_\infty ||b_\beta||_\infty \Bigg)$$

Finally, we prove that $||v^s_\beta - v^s_{1-\lambda}||_\infty \xrightarrow[\beta \to (1-\lambda)^-]{} 0$. Recall notations from Proposition 3.1 and let V be a neighbourhood of some $\sigma \in \Sigma$ as in the proof of Theorem 3.4. Let $x \in f^{-N}(V) \cap U_\sigma$ and let $n(x)$ be the number of points in the orbit of x that belong to $B_{\sigma,\delta} \setminus V$. Then $N - n(x) \geqslant 0$ and we have the following estimates depending on i:

- if $i \leqslant N - n(x)$, then $|s_i(x)| \leqslant \lambda^{-i} \left(\frac{1}{1+\eta}\right)^{i+1} \sup |b|$.
- if $N - n(x) < i \leqslant N$, then $|s_i(x)| \leqslant \lambda^{-i} \left(\frac{1}{1+\eta}\right)^{N-n(x)} \left(\frac{1}{\lambda+\beta}\right)^{i-(N-n(x))} \sup |b|$

so that $|s_i(x)| \leqslant \sup |b| \lambda^{-i} \left(\frac{1}{\lambda+\beta}\right)^i$.

- if $i = j + N > N$, then $|s_i(x)| \leqslant \lambda^{-(j+N)} \left(\frac{1}{\lambda+\beta}\right)^{j+N} \mathrm{Lip}(b) \varepsilon \max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma)^j$.

Therefore, $\sum_{i \geqslant 0} |s_i(x)| \leqslant ||b||_\infty \left(\frac{\lambda}{\lambda-1} + \frac{\lambda(\lambda+\beta)}{\lambda(\lambda+\beta)-1} + \varepsilon \frac{\mathrm{Lip}(b)}{1 - \frac{\max(\lambda^{-1}, \lambda+\beta+\delta)}{\lambda(\lambda+\beta)}} \right)$, and so for all $\varepsilon > 0$.

Hence, the family of vector fields $(v^s_\beta)_\beta$ is uniformly bounded on S_g and the bound can be chosen uniformly in β for $\beta \in [1-\lambda-\varepsilon, 1-\lambda]$. However, the estimates we had on the Lipschitz constants are no longer good enough to apply the same argument as in above cases.

Let $x \in S_g$ and $(x_n, \beta_n)_n$ be a sequence converging to $(x, 1-\lambda)$ and such that $\beta_n < 1-\lambda$ for all n. For n large enough, the sequence $(v^s_{\beta_n}(x_n))_n$ is bounded and let

$w(x)$ be a sub-sequential limit. Since for all $k \geqslant 0$, the sequence $(v^s_{\beta_n}(f^k_{\beta_n}(x_n)))_n$ is bounded, by a diagonal argument we can assume up to extracting that the sequences converge to some vectors $w(f^k_{1-\lambda}(x))$. By continuity of $\mathrm{d}f_\beta$ in β, we get that $\mathrm{d}_x f^k_{1-\lambda} w(x) = \lambda^{-k} w(f^k_{1-\lambda}(x))$ for all k. By expressing vectors $w(f^k_{1-\lambda}(x))$ in the basis $(v^s_{1-\lambda}(x), e_v)$, we see that $w(x) \in \mathbb{R} v^s_{1-\lambda}(x)$. Since each vector of the form $v^s_{\beta_n}(f^k_{\beta_n}(x_n))$ has a component equal to 1 along e_v in the basis (e_h, e_v), we get $w(x) = v^s_{1-\lambda}(x)$. Hence $(x, \beta) \mapsto v^s_\beta(x)$ is continuous at $(x, (1-\lambda)^-)$.

Now, suppose that $||v^s_\beta - v^s_{1-\lambda}||_\infty$ does not converge to zero as β converges to $1-\lambda$ from below. Then, there exists some positive ε and sequences $(\beta_n)_n$ and $(x_n)_n$ such that $\lim\limits_{n\to\infty} \beta_n = (1-\lambda)^-$ and $||v^s_{\beta_n}(x_n) - v^s_{1-\lambda}(x_n)|| \geqslant \varepsilon$. Up to extracting, we can assume that $(x_n)_n$ converges to some x. Therefore $||v^s_{\beta_n}(x_n) - v^s_{1-\lambda}(x)|| \geqslant \varepsilon/2$ for large enough n. This contradicts the continuity of $(x, \beta) \mapsto v^s_\beta(x)$ at $(x, (1-\lambda)^-)$.

The continuity of $(x, \beta) \mapsto v^s_\beta(x)$ on $(S_g \smallsetminus \Sigma) \times] - \lambda + \lambda^2, 0]$ follows from

$$||v^s_\beta(x) - v^s_{\beta_0}(x_0)|| \leqslant ||v^s_\beta - v^s_{\beta_0}||_\infty + ||v^s_{\beta_0}(x) - v^s_{\beta_0}(x_0)|| \xrightarrow[(x,\beta)\to(x_0,\beta_0)]{} 0.$$

□

3.5. **Renormalized flow and topological properties of K.** Since v^s is Lipschitz continuous, we can integrate it into a flow h_t. Since some trajectories reaches in finite time conical points, for which v^s is not defined, this flow must be treated carefully. On the other hand, since v^s is uniformly contracted by the action of f, h_t is renormalized by f. From this relationship between f and h_t, we can deduce further topological properties about stable leaves and the set K. We first prove that for each fixed hyperbolic point p^σ_i, its stable leaf coincides with the orbit by h_t of this point. From this fact and Proposition 2.8, we deduce that K is transverse to any vertical leaf. We then show that K is in fact equal to the closure of the stable leaf of any hyperbolic fixed point p^σ_i, hence K is connected. Finally, we prove that f is topologically transitive with respect to the trace topology of S_g on K.

Proposition 3.12. *For all $x \in S_g \smallsetminus \Sigma$ and t for which $h_t(f(x))$ is well defined, f and h_t satisfy the relation,*

$$f \circ h_{\lambda t}(x) = h_t \circ f(x).$$

Proof. Since $\mathrm{d}_x f(v^s(x)) = \lambda^{-1} v^s(f(x))$, for $x \in S_g \smallsetminus \Sigma$, notice that,

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ h_{\lambda t}(x)) = \mathrm{d}_{h_{\lambda t}(x)} f \left(\frac{\mathrm{d}}{\mathrm{d}t} h_{\lambda t}(x) \right) = \mathrm{d}_{h_{\lambda t}(x)} f(\lambda v^s(h_{\lambda t}(x))) = v^s(f \circ h_{\lambda t}(x)).$$

Therefore the two functions $t \to f(h_{\lambda t}(x))$ and $t \to h_t(f(x))$ solve the same differential problem with the same initial condition. Hence $f \circ h_{\lambda t} = h_t \circ f$ for all t where the solution is defined. □

This commutation relation between f and h_t is a central argument throughout the rest of this article. First, it is used to prove that the set K is invariant by the flow. To this end, we have to verify that the flow is complete on K.

Let $\mathcal{F} := S_g \smallsetminus (\Sigma \cup \{x \in S_g \smallsetminus \Sigma \mid \forall t \in \mathbb{R},\ h_t(x) \text{ exists}\})$ be the set of points whose trajectory are not well defined for all time. We can fully caracterise this set, this is the subject of the following lemma.

Lemma 3.13. *If $x \in \mathcal{F}$, then there exist $\sigma \in \Sigma$ and $t_0 \in \mathbb{R}$ such that $h_t(x) \to \sigma$ as t tends to t_0*

Proof. By compactness of S_g, up to taking a sub-sequence $(t_n)_n$ that converges to t_0, the limit of $(h_{t_n}(x))_n$ exists. If this limit doesn't belong to Σ, we can extend the solution past t_0. □

Proposition 3.14. *The orbit $\{h_t(x)\}$ of any point x in K is well defined for all time t. Furthermore, for all $t \in \mathbb{R}$, $h_t(K) = K$.*

Proof. We prove that $\mathcal{F} \subset U$. By contradiction, let $x \in \mathcal{F} \cap K$. Let t_0 and σ be as in Lemma 3.13. Hence, the smooth curves $f^n \circ h_t(x) : t \in [0, t_o] \to S_g$ join K to Σ and their lengths are less than $\lambda^{-n} t_0 ||v^s||_\infty$. This contradicts the fact that $\mathrm{d}(K, \Sigma) > \min\{|p^\sigma| \mid \sigma \in \Sigma\} > 0$ by Proposition 2.4.

Since $\mathcal{F} \cap K = \emptyset$, $h_t(x)$ is well defined for all $x \in K$ and all time t. Let $x \in K$. By contradiction, assume there exists t_1 such that $h_{t_1}(x) \in U$. Therefore $f^n(h_{t_1}(x))$ converges to some σ as n goes to infinity and the curves $f^n \circ h_t(x) : t \in [0, t_1] \to S_g$ joins K to some arbitrarily close point to σ for n large enough. Since such a curve is of length at most $\lambda^{-n} t ||v^s||_\infty$, it contradicts $\mathrm{d}(K, \Sigma) > 0$. □

We can now deduce the announced topological properties of the invariant leaves and of K.

Proposition 3.15. *For all p_i^σ, we have the equality of sets $W^{ss}(p_i^\sigma) = h_{\mathbb{R}}(p_i^\sigma)$.*

Proof. Let $t \in \mathbb{R}$. Hence $f^n(h_t(p_i^\sigma)) = h_{\lambda^{-n} t}(p_i^\sigma)$ converges to p_i^σ as n goes to infinity. Hence $h_{\mathbb{R}}(p_i^\sigma) \subset W^{ss}(p_i^\sigma)$. By the commutation relation between f and h_t, we get that $h_{\mathbb{R}}(p_i^\sigma)$ is invariant by f. In the linearisation near p_i^σ given by the Grobman–Hartman theorem, the only invariant part by f corresponds to a small piece γ of the stable leaf of p_i^σ. By invariance of $h_{\mathbb{R}}(p_i^\sigma)$ by f, we get $\gamma \subset h_{\mathbb{R}}(p_i^\sigma)$. Finally, since $W^{ss}(p_i^\sigma) = \bigcup_{n \geqslant 0} f^{-n}(\gamma)$, we get $h_{\mathbb{R}}(p_i^\sigma) = W^{ss}(p_i^\sigma)$. □

Corollary 3.16. *The set K is transverse to any vertical leaf.*

Proof. Since the convergence of the infinite sum defining v^s is uniform on K, the vertical component of the vector field v^s is continuous, hence bounded. Therefore, all the stable leaves $W^{ss}(p_i^\sigma)$ are transverse to any vertical leaf. The result holds by taking the closure since slopes are bounded and by Proposition 2.8. □

Theorem 3.17. *The set K is connected and it can be written as $K = \overline{W^{ss}(p_i^\sigma)}$, for any $\sigma \in \Sigma$ and any $1 \leqslant i \leqslant 2n_\sigma$.*

Proof. Let $\sigma_1, \sigma_2 \in \Sigma$ and i_1, i_2 be two integers. For simplicity, call $p_1 = p_{i_1}^{\sigma_1}$ and $p_2 = p_{i_2}^{\sigma_2}$. Let W_2 be the open set given by the Grobman–Hartman theorem – without loss of generality we assume it is a rectangle with horizontal and vertical sides. Since $W^{su}(p_2)$ contains a dense vertical leaf, and $W^{ss}(p_1)$ is transverse with all vertical leaves, the intersection $W^{su}(p_2) \cap W^{ss}(p_1)$ is non-empty. Let $x \in W^{su}(p_2) \cap W^{ss}(p_1)$ and let γ be a small connected piece of $W^{ss}(p_1)$ containing x in its interior. Then, for large enough $n \geqslant 0$, we see that $f^{-n}(\gamma) \cap W_2$ accumulates on $W^{ss}(p_2) \cap W_2$. Therefore, $W^{ss}(p_2) \cap W_2 \subset \overline{W^{su}(p_2) \cap W^{ss}(p_1)} \subset \overline{W^{ss}(p_1)}$. Since $\overline{W^{ss}(p_1)}$ is invariant by the action of f and $W^{ss}(p_2) = \bigcup\limits_{n \geqslant 0} f^{-n}(W^{ss}(p_2) \cap W_2)$, we get the inclusion $\overline{W^{ss}(p_2)} \subset \overline{W^{ss}(p_1)}$. Since the choice of p_1 and p_2 is arbitrary, the result follows from Proposition 2.8. □

Theorem 3.18. *The function $f : K \to K$ is transitive with respect to the trace topology of S_g on K.*

Proof. Let U_1 and U_2 be open sets in S_g that have non-empty intersection with K. Let $p_1 = p_{i_1}^{\sigma_1}$ and $p_2 = p_{i_2}^{\sigma_2}$ for some σ_1, $\sigma_2 \in \Sigma$ such that $U_i \cap (W^{ss}(p_i) \cap W^{su}(p_i)) \neq \emptyset$ for $i = 1, 2$. Since $W^{ss}(p_2)$ is transverse with all the vertical leaves, we can find a rectangle V_2 contained in U_2 whose sides are vertical and horizontal, such that $W^{ss}(p_2)$ crosses V_2 *from side to side*.

By density of $W^{su}(p_1)$, there exists $x_2 \in V_2 \cap W^{su}(p_1)$. Let W_1 be the open set of linearisation near p_1 – without loss of generality, we can assume W_1 to be a rectangle with horizontal and vertical sides. For large enough $n \geqslant 0$, the set $f^{-n}(V_2)$ crosses *horizontally* W_1.

Let $x_1 \in U_1 \cap W^{ss}(p_1)$ and $\varepsilon > 0$ be such that the vertical segment γ of length ε, containing x_1 in its interior, is contained in U_1. For all large enough $m \geqslant 0$, the line $f^m(\gamma)$ crosses *vertically* W_1. Hence $f^m(U_1) \cap f^{-n}(U_2) \neq \emptyset$. □

It easily follows from the transitivity of f and the closing lemma that periodic points of f are dense in K. Therefore K is an Axiom A attractor in the sense of [24].

Theorem 3.19. *If Σ^ε is an open ε-neighbourhood of Σ for some small enough $\varepsilon > 0$, $U \coloneqq S_g \smallsetminus \overline{\Sigma^\varepsilon}$ and f^{-1} is $\mathcal{C}^2$ away from Σ, then K is an Axiom A attractor for $f^{-1} : U \to U$.*

4. Each f is mixing

In this part we prove that there exists a measure with respect to which f is mixing. To do so, we first claim that the unique ergodicity of h_t is a sufficient condition for the mixing f and that f is mixing with respect to the unique invariant measure of h_t. We then prove that h_t is indeed uniquely ergodic, with the support of its unique invariant measure being exactly K.

Theorem 4.1. *The flow h_t is uniquely ergodic. Furthermore the support of the invariant measure is K.*

Corollary 4.2. *The unique invariant measure μ of h_t is also invariant by f and f is mixing with respect to μ.*

This corollary is a partial restatement of Theorem 1.1.

Proof. Since K is invariant by f and by the flow h_t and since h_t is well defined for all t on K, we have

$$f_*\mu = f_*((h_t)_*\mu) = (f \circ h_t)_*\mu = (h_{\lambda^{-1}t})_*(f_*\mu).$$

Therefore the measure $f_*\mu$ is invariant by the flow h_t. By unique ergodicity of the flow, we must have $f_*\mu = \mu$.

Let $F \in \mathrm{L}^2(\mu)$ be such that $\int F \,\mathrm{d}\mu = 0$. We now prove that the sequence $(F \circ f^n)_n$ weakly converges to zero. By invariance of the measure, the sequence is bounded in the $\mathrm{L}^2(\mu)$ norm. By the Banach-Alaoglu-Bourbaki theorem, this sequence lives in a weakly compact set. Let $\bar{F}$ be a sub-sequential weak limit of $(F \circ f^n)_n$ and let $(n_k)_k$ be a strictly increasing sequence of integers such that

$F \circ f^{n_k} \xrightarrow[k\to\infty]{} \bar{F}$. On the other hand,

$$\begin{aligned} ||F \circ f^{n_k} \circ h_t - F \circ f^{n_k}||_{L^2} &= ||F \circ h_{\lambda^{-n_k}t} \circ f^{n_k} - F \circ f^{n_k}||_{L^2}, \\ &= ||F \circ h_{\lambda^{-n_k}t} - F||_{L^2} \xrightarrow[k\to\infty]{} 0, \end{aligned}$$

where the final limit follows from the density of continuous functions in $\mathrm{L}^2(\mu)$. Now, $F \circ f^{n_k} \circ h_t - F \circ f^{n_k}$ converges weakly to $\bar{F} \circ h_t - \bar{F}$. The identification of the strong limit with the weak limit gives $\bar{F} \circ h_t - \bar{F} = 0$. By unique ergodicity of $(h_t)_t$, $\bar{F}$ is constant. By integration, this constant is zero. Hence all the sub-sequential weak limit of $(F \circ f^n)_n$ are 0, which proves the mixing. □

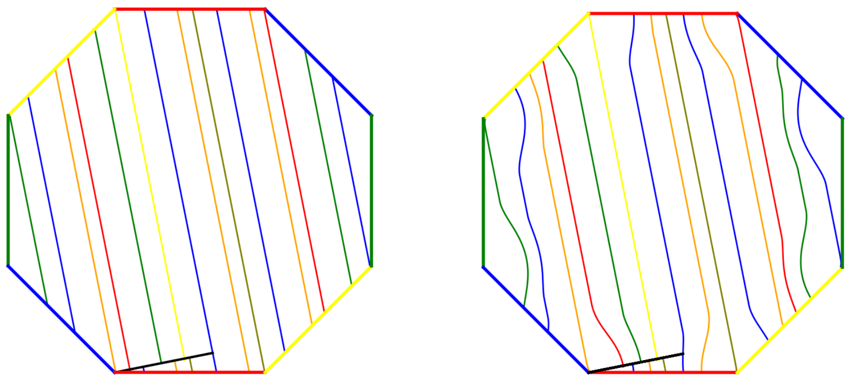
In order to prove Theorem 4.1, we heavily rely on the semi-conjugacy theorem from [27, Proposition 7], more precisely if an Interval Exchange Transformation (IET) and a Generalized Interval Exchange Transformation (GIET) have the same combinatorial datum and follow a same full path in the Rauzy diagram – when renormalized by the Rauzy–Veech algorithm – then there exists a continuous, increasing and surjective function that semi-conjugates the two transformations.

4.1. **Construction of a GIET and h_t as its suspension flow.** Recall some notation from Section 2.1. Let φ be the pseudo-Anosov map that we perturbed in order to get f. By construction, φ fixes each conical point and each separatrix. Let $\sigma \in \Sigma$ be a conical point and γ_0 be a segment of a vertical separatrix starting at σ such that $\sigma \in \partial\gamma_0$. From a general property of the pseudo-Anosov maps (see [19, proposition 5.3.4]), there exists a decomposition in rectangles $\mathcal{R}_0 = (R_1^0, \dots, R_{|\sigma|}^0)$ of S such that (up to shortening γ_0) the *bases* of these rectangles form a partition of γ_0.

Denote by $\partial_v\mathcal{R}_0$ (resp. $\partial_h\mathcal{R}_0$) the vertical (resp. horizontal) components of $\bigcup_i \partial R_i$. By construction, $\partial_h\mathcal{R}_0 = \gamma_0$. Now, $\partial_v\mathcal{R}_0$ is made of portions of trajectories for the horizontal flow associate to φ that connect a conical point to γ_0, but don't intersect γ_0 at some other previous time.

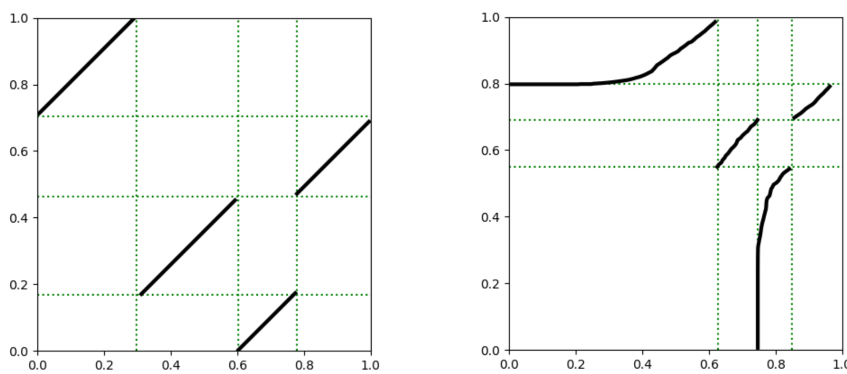
Since the family of vector fields $(x, \beta) \mapsto v_\beta^s(x)$ is continuous, we can deform by some homotopy $\mathcal{R}_0$ into $\mathcal{R}_\beta = (R_0^\beta, \dots, R_{|\Sigma|}^\beta)$ while preserving the vertical direction, where β is the amplitude of the perturbations in the construction of f. In more details, the homotopy sends the portions of trajectories of the horizontal flow that connect conical points to γ_0, to the portions of trajectories of h_t which contain a conical point. Since the vector field v_β^s has its horizontal component constant equal to 1, these latter trajectories are the ones connecting conical points to γ, where γ is a slightly longer or shorter copy of γ_0. Since any two trajectories do not intersect, these portions of trajectories of h_t are still the shortest ones that connect conical points to γ

Call T (resp. T_0) the Poincaré first return map to γ (resp γ_0) of h_t (resp. of the unit speed horizontal flow associate to φ). It is clear from the construction that T_0 is an IET and that T is a GIET. Since for all β, the horizontal component of v_β^s is equal to one, both T and T_0 have the same combinatorial data. Furthermore, by construction, T and T_0 have the same path in the Rauzy-graph – otherwise for some parameter β^* the GIET T_{β^*} induced by $\mathcal{R}_{\beta^*}$ would have a connection, which corresponds geometrically to a side of a rectangle of $\mathcal{R}_{\beta^*}$ connecting a conical point to another one: this is impossible since f_{β^*} would contract this curve.



Rectangle decomposition $\mathcal{R}_0$ Perturbed rectangle decomposition $\mathcal{R}_\beta$

FIGURE 2. Rectangle decompositions in the case of a flat genus two surface. The pseudo-Anosov transformation on this surface is explicited in the appendix of [25] as the composition of an upper triangular matrix with its transpose matrix.



Graph of the IET associated to $\mathcal{R}_0$ Graph of the GIET associated to $\mathcal{R}_\beta$

FIGURE 3. Graphs on the induced IET and GIET induced respectively by the rectangle decompositions in Figure 2 – both flows are going *"downward"*.

Since foliations associated to a pseudo-Anosov have no closed leaf (see [13]), it follows that T_0 has no connection, hence, by [27], the path of T_0 in the Rauzy graph is full and so T_0 and T are semi-conjugated by a continuous, increasing and surjective function. Also, since T_0 has no connection, it is minimal.

We summarize all this in the following proposition.

Proposition 4.3. *If σ is a conical point, there exist two portions of a same separatrix (both containing σ) γ_0 and γ, and maps $T : \gamma \to \gamma$, $T_0 : \gamma_0 \to \gamma_0$ such that:*

(i) T_0 is an IET and T is a GIET,
(ii) T_0 is the Poincaré first return map of the horizontal flow associated to φ,
(iii) T is the Poincaré first return map of the flow h_t associated to f,
(iv) T_0 and T have the same combinatorial data, and the same path in the Rauzy-graph,
(v) there exists a continuous, increasing and surjective function h such that $h \circ T = T_0 \circ h$.

4.2. **Minimality of the flow on K.** In this part we prove that the map T – from which h_t is the suspension flow – is minimal on its nonwandering set. To do so, we rely on the analysis carried out in [27]. From this, we deduce that the flow h_t acts minimally on K – actually, we also prove that K is an attractor for positive and negative times. This property will be useful to prove that the support of the unique invariant measure of h_t is K.

As in [27], define $S(\infty)$ as the union of the forward orbit of the discontinuity points of T^{-1} and the backward orbits of the discontinuity points of T. Similarly, define $S_0(\infty)$ from the discontinuities of T_0 and T_0^{-1}. By construction, h is an increasing bijection from $S(\infty)$ to $S_0(\infty)$.

Define Ω as the set of non-isolated points of $\overline{S(\infty)}$. Clearly, Ω is a closed set. In order to prove that T is minimal on Ω, we first need the following lemma.

Lemma 4.4. *There exists a decomposition of Ω in closed sets $\Omega = \Omega_+ \cup \Omega_-$ such that $T(\Omega_+) \subset \Omega_+$ and $T^{-1}(\Omega_-) \subset \Omega_-$.*

Proof. Let $S(\infty)_+$ be the forward orbits by T of the discontinuity points of T^{-1} and similarly $S(\infty)_-$ be the set of the backward orbits by T of the discontinuity points of T. By definition of $S(\infty)$, $S(\infty) = S(\infty)_+ \cup S(\infty)_+$. Define $\Omega_\pm$ as the set of non-isolated points of $\overline{S(\infty)_\pm}$. These sets satisfy the conclusion of the lemma. □

Theorem 4.5. *When restricted to the set Ω, T is minimal.*

Proof. Let x be a point of Ω. Up to considering its backward orbit, we assume that $x \in \Omega_+$. We want to prove that $(T^n(x))_{n\geqslant 0}$ is dense in Ω. By contradiction, let U be an open set such that $U \cap \Omega \neq \emptyset$ and $T^n(x) \notin U \cap \Omega$ for all n. Since Ω_+ is stable by the action of T, we can relax the last condition by $T^n(x) \notin U$ for all $n \geqslant 0$.

Since $U \cap \Omega \neq \emptyset$, $U \cap \Omega$ contains at least two different points of $S(\infty)$, therefore h is not constant on U. Hence $h(U)$ has a non-empty interior. Finally, since the sequence $h \circ T^n(x) = T_0^n(h(x))$ avoids an open set and T_0 is minimal, we get a contradiction. □

In order to prove that Ω is an attractor for both T and T^{-1}, we need the following three technical lemmas.

Lemma 4.6. *The function h such that $h \circ T = T_0 \circ h$ is constant on the connected components of $\gamma \setminus \overline{S(\infty)}$.*

Proof. By contradiction, let $]j_-, j_+[$ be a connected component of $\gamma \setminus \overline{S(\infty)}$ on which h is not constant. Therefore $h(j_-) < h(j_+)$. By density of $S_0(\infty)$ in γ_0, there exist infinitely many points of $S_0(\infty)$ in the middle third segment of $[h(j_-), h(j_+)]$. Since

$h : S(\infty) \to S_0(\infty)$ is a bijection, the image by h^{-1} of all these points of $S_0(\infty)$ is relatively compact in $]j_-, j_+[$. Hence, there exist accumulation points of $S(\infty)$ in $]j_-, j_+[$, which is a contradiction. □

Lemma 4.7. *The connected components of $\gamma \setminus \overline{S(\infty)}$ are permuted without cycle by T.*

Proof. By construction of $S(\infty)$, T and T^{-1} are continuous on each connected componant of $\gamma \setminus \overline{S(\infty)}$. If J is a connected componant of $\gamma \setminus \overline{S(\infty)}$, then it is easy to see that $T(J)$ is a subset of a connected componant of $\gamma \setminus \overline{S(\infty)}$. The same argument applied with T^{-1} proves that the connected componants are permuted by the action of T.

By contradiction, let J be a connected component of $\gamma \setminus \overline{S(\infty)}$ and $n > 0$ be such that $T^n J = J$. Therefore $h \circ T^n(J) = h(J) = \{x\}$ by the Lemma 4.6. Now $h \circ T^n(J) = T_0^n(h(J))$. Therefore x is a periodic point for T_0, which contradicts the minimality of T_0. □

Lemma 4.8. *The isolated points of $\overline{S(\infty)}$ are wandering points.*

Proof. Let x be an isolated point of $\overline{S(\infty)}$. Therefore there exists an open set U such that $U \cap \overline{S(\infty)} = \{x\}$. Hence $U \setminus \{x\} = U_1 \sqcup U_2$ is included in the union of two connected components of $\gamma \setminus \overline{S(\infty)}$, which are wandering sets by lemma 4.7. Therefore, $T^n(U \setminus \{x\}) \cap U \neq \emptyset$ for only finitely many values of n. Now, if $T^n(x) \in U$ then $T^n(x) = x$ and therefore $h(x)$ is a periodic point of T_0 which is impossible. Finally, we proved that $T^n U \cap U \neq \emptyset$ for only finitely many values of n, in other words x is a wandering point. □

Theorem 4.9. *For every point $x \in \gamma$ whose forward orbit is infinite, then the ω-limit set of x satisfies $\omega(x) = \Omega$. The counterpart is true for infinite backward orbits and α-limit sets. In other words, Ω is an attractor for the transformations T and T^{-1}.*

Proof. We prove both inclusions. We start by showing that $\Omega \subset \omega(x)$. By contradiction, let $y \in \Omega$ such that $y \notin \omega(x)$. Since $\omega(x)$ is a closed set, there exists an open set U containing y such that $U \cap \Omega \neq \emptyset$ and $U \cap \omega(x) = \emptyset$. Therefore $T^n(x) \notin U$ for large enough n. Since $U \cap \Omega \neq \emptyset$, U contains at least two distinct points of $S(\infty)$. Since h is one-to-one on $S(\infty)$ and continuous on γ, the set $h(U)$ has a non-empty interior. Therefore the sequence $T_0^n(h(x)) = h \circ T^n(x)$ is dense in γ_0 (by minimality of T_0) and avoids the set of non-empty interior $h(U)$, hence a contradiction.

We now prove that $\Omega^c \subset \omega(x)^c$. Let y be in Ω^c. There are two cases. If $y \in \gamma \setminus \overline{S(\infty)}$, then by Lemma 4.7 y is contained in a wandering interval: y cannot be obtain as a limit point of an orbit by T, hence $y \notin \omega(x)$. Otherwise, y is an isolated point of $\overline{S(\infty)}$. By contradiction, $y \in \omega(x)$ implies that y is a non-wandering point, which contradicts Lemma 4.8. □

Theorem 4.10. *The set Ω coincide with the non-wandering set $\Omega(T)$ of T.*

Proof. By minimality of T when restricted to Ω, we get $\Omega \subset \Omega(T)$. Since T permutes the connected components of $\gamma \setminus \overline{S(\infty)}$, all points of $\gamma \setminus \overline{S(\infty)}$ are wandering points. Therefore $\Omega(T) \subset \overline{S(\infty)}$. Finally, by the Lemma 4.8 we can refined this last inclusion by $\Omega(T) \subset \Omega$. □

Proposition 4.11. *The sets Ω and K are related by $\Omega = \gamma \cap K$.*

Proof. Let $p = p_i^\sigma$ be in $\gamma \cap K$. We know that $h_{\mathbb{R}}(p)$ is dense in K, therefore $(T^n(p))_n$ is dense in $\gamma \cap K$. However, $\omega_T(x) = \Omega$ for all $x \in \gamma$, in particular for $x = p$. Hence $\Omega = \gamma \cap K$. □

Theorem 4.12. *When restricted to K, the flow h_t is minimal. Furthermore, the set K is an attractor for the flow h_t, for positive and negative times.*

Proof. Let $u : \gamma \to \mathbb{R}$ be the function giving the first return time in γ. This function is bounded by some constant C. Clearly, we have the equality $h_{\mathbb{R}}(\Omega) = h_{[0,C]}(\Omega)$ and the left hand side is a closed set containing the orbit of $p = p_i^\sigma \in \gamma \cap K = \Omega$, hence $h_{[0,C]}(\Omega) = K$. This last equality proves the minimality of $(h_t)_t$ when restricted to K.

From $h_{[0,C]}(\Omega) = K$ and Theorem 4.9, we obtain that every infinite forward trajectory of h_t accumulates on K. Similarly, every infinite backward trajectory of h_t accumulates on K. □

4.3. **Proof of the unique ergodicity of h_t.**

Lemma 4.13. *In the coordinates of the suspension, every h_t-invariant measure μ must be of the form $\mathrm{d}\mu(x,t) = C\,\mathrm{d}\nu(x)\,\mathrm{d}Leb(t)$, for $x \in \gamma$, $0 \leqslant t < u(x)$, some constant $C > 0$ and some measure ν on γ, where $u(x)$ is the time of first return to γ of x and Leb is the Lebesgue measure.*

Proof. Let $\tilde{\pi} : \gamma \times \mathbb{R} \to \mathcal{R}$ be a covering map. The lift of h_t is simply the unit speed translation flow along the second coordinate. Let μ be an invariant measure for this flow. Let $\tilde{\mu}$ be a lift of μ to $\gamma \times \mathbb{R}$. Therefore $\tilde{\mu}$ is invariant by translation along the second coordinate. Hence $\tilde{\mu} = C\nu \otimes Leb$, where Leb is the Lebesgue measure and $\nu(S) := \tilde{\mu}(S \times [0,\varepsilon])$ is a measure on γ, for some $\varepsilon > 0$. Taking back the projection by $\tilde{\pi}$, we get $\mathrm{d}\mu(x,t) = C\,\mathrm{d}\nu(x)\,\mathrm{d}Leb(t)$, as long as $\varepsilon < \inf_x u(x)$. □

We can now prove the unique ergodicity of h_t.

Proof of theorem 4.1. Let μ be a measure invariant by the flow h_t. By Lemma 4.13, we can find a constant C and a measure ν on γ such that $\mathrm{d}\mu(x,t) = C\,\mathrm{d}\nu(x)\,\mathrm{d}Leb(t)$. By applying Fubini's theorem on sufficiently small rectangles, we obtain that ν is invariant by T.

Since the horizontal foliation associated to a pseudo-Anosov map is uniquely ergodic – see [14, Exposé 12] – it follows that T_0 is uniquely ergodic.

Now, T and T_0 have the same path in the Rauzy-graph. By [27], T is semi-conjugated to T_0 by some continuous monotonic function h. This function h is bijective when restricted, up to a countable set of points, to the set of non-wandering points of T. Therefore T is also uniquely ergodic, of invariant measure ν.

Hence h_t is uniquely ergodic, of invariant measure μ.

We now prove that the support of μ is K. First, since $supp(\nu)$ is included in the set of non-wandering points of T, which is Ω, and $supp(\nu)$ is a closed set invariant by T, by minimality of T we get that $supp(\nu) = \Omega$. Now, by the factorization of μ and the fact that $h_{\mathbb{R}}(\Omega) = K$, we get $supp(\mu) = K$. □

As a final remark for this section, we can perform a similar analysis by pertubing a pseudo-Anosov only at some conical points $\Sigma_0 \subsetneq \Sigma$. The proofs are mostly the same by replacing Σ by Σ_0.

We give in Figure 4 a graphical representation of the set K in the case of the fully explit example outlined in the description of Figure 2.

5. Perturbation at a regular periodic point

Because of the following general property concerning pseudo-Anosov maps – see for example [13] – we can consider periodic points that are not conical points – they are regular points.

Proposition 5.1. *If $\varphi : S_g \to S_g$, then the set of periodic points of φ is a dense subset of S_g.*

Let $\theta \in S_g \setminus \Sigma$ be a periodic point of φ that is not a conical point. Up to considering a power of φ, we assume that θ is a fixed point.

In this part, we present that a very similar analysis can be done when a pseudo-Anosov map is perturbed at a fixed point that is regular instead of conical.

5.1. Definitions, regularity and first properties.

We can proceed to the same type of perturbation as described in Section 2.1 at a regular fixed point θ, except that it is much easier to define since θ is not a conical point and we do not have to deal with branched cover.

Write $\varphi(x+iy) = \lambda x + i\lambda^{-1}y$ in some local chart centred at θ. In these coordinates, define

$$f(x+iy) \coloneqq \left(\lambda + \beta k\left(\frac{x^2+y^2}{\alpha}\right)\right) x + i\lambda^{-1}y,$$

for some $\beta \in]-\lambda, -\lambda+1[$, $0 < \alpha < \min\left(\frac{1}{2}Syst(S_g), \inf\{\mathrm{d}(\theta,\sigma) \mid \sigma \in \Sigma\}\right)$ and $k : \mathbb{R} \to \mathbb{R}$ is an even unimodal function of class $\mathcal{C}^1$, compactly supported in $[-1,1]$ such that k' is Lipschitz continuous, for example $k(r) = (1-r^2)^2\mathbb{1}_{[-1,1]}$. Set $f = \varphi$ elsewhere. With these parameters, f is regular in the following way.

Proposition 5.2. *If $\beta \in]-\lambda, 0]$ and $0 < \alpha < \min\left(\frac{1}{2}Syst(S_g), \inf\{\mathrm{d}(\theta,\sigma) \mid \sigma \in \Sigma\}\right)$, then f is a homeomorphism on S_g and is a diffeomorphism on $S_g \setminus \Sigma$.*

Also, for a refined condition on β, we get

Proposition 5.3. *If $\beta \in]-\lambda, -\lambda+1[$ and $0 < \alpha < \min\left(\frac{1}{2}Syst(S_g), \inf\{\mathrm{d}(\theta,\sigma) \mid \sigma \in \Sigma\}\right)$, then θ is an attractive fixed point for f. Call U_θ its basin of attraction. Moreover U_θ is an open set.*

Define $K \coloneqq S_g \setminus U_\theta$ to be the complement of the basin of attraction of θ. Clearly, K is a compact subset, invariant by f.

Our goal is to understand the dynamical behaviour of f on K – and near it. First we need to give some more topological properties about the set K. The next property also shows that K is not the empty set.

Proposition 5.4. *If $\beta \in]-\lambda, 1-\lambda[$ and $\alpha < \delta_\Sigma/2$, then there exist fixed hyperbolic points p_i, $i \in \{1,2\}$, one on each vertical ray starting at θ. These two points are at the same distance $|p|$ from θ. Furthermore $B(\theta, |p|) \subset U_\theta$.*

All the proofs of these properties are essentially the same as in Subsection 2.2. In fact all the following properties are proved by very similar arguments – if not the same – as their counterparts in the previous case of a perturbation at a conical point.

Proposition 5.5. *The following properties, similar to the case studied in Sections and 2 and 3, hold.*

(i) the open set U_θ is dense in S_g.
(ii) The set K is hyperbolic. The stable vector field v^s is given by the same formula as in Theorem 2.9.
(iii) The formula giving v^s on K still makes sense on $S_g \setminus \Sigma$, and defines a bounded Lipschitz continuous vector field, still noted v^s, when $\beta \in]-\lambda+\lambda^{-2}, 0]$.
(iv) The flow h_t generated by v^s satisfies $f \circ h_{\lambda t} = h_t \circ f$ whenever both sides are well defined.
(v) The set K is invariant by h_t.
(vi) The set K is the closure of the trajectory of p_i, $i \in \{1, 2\}$, under h_t, in fact $K = \overline{W^{ss}(p_i) \cap W^{su}(p_i)}$. Furthermore $W^{ss}(p_i) = h_{\mathbb{R}}(p_i)$, and so K is connected.

5.2. **Finer properties about dynamics of f and h_t.** Again, with almost the same proof as Theorem 3.10, we can prove that

Proposition 5.6. *The vector field $v^s = v^s_\beta$ depends on β since $f = f_\beta$ does. Furthermore, the map $(x, \beta) \mapsto v^s_\beta(x)$ is continuous on $(S_g \setminus \Sigma)\times] - \lambda + \lambda^{-2}, 0]$.*

From this last property, we can construct a rectangle decomposition of S_g in a similar manner as previously. This time, the segment γ will start at θ. In order to construct such a decomposition $\mathcal{R}$, we start from a decomposition $\mathcal{R}_0$ – with *straights* rectangles – associated to the horizontal flow and to a segment γ_0 starting at θ and included in a vertical leaf. To get $\mathcal{R}$ we deform $\mathcal{R}_0$ as described in 4.1. These decompositions lead to the following proposition.

Proposition 5.7. *Similarly to Section 4, the following properties hold.*

(i) The flow h_t induces a map $T : \gamma \to \gamma$, which is the Poincaré first return map of this flow. The map T is a GIET. By construction, h_t can be recovered by taking a suspension flow over T.
(ii) The horizontal (unit speed) flow induces a map $T_0 : \gamma_0 \to \gamma_0$, which is the Poincaré first return map of this flow. The map T_0 is an IET. By construction, the horizontal flow can be recovered by taking a suspension flow over T_0.
(iii) The maps T and T_0 have the same path in the Rauzy diagram. Furthermore this path is full. Hence T is semi-conjugated to T_0.

In a very similar fashion as in Subsections 4.2 and 4.3, since T_0 is minimal and uniquely ergodic, we can prove the unique ergodicity of h_t and its minimality when restricted to K. We sum up these results in the following theorem.

Theorem 5.8. *As in Section 4, the dynamic of f and h_t satisfies the following properties.*

(i) For every x in $S_g \setminus \Sigma$ such that its forward trajectory by h_t is defined for all times, its ω-limit set coincides with K, $\omega(x) = K$. The same goes for backward trajectories and α-limit sets.
(ii) The flow h_t is uniquely ergodic, of unique invariant measure noted μ. By uniqueness and the commutation property between h_t and f, μ is also invariant by f.

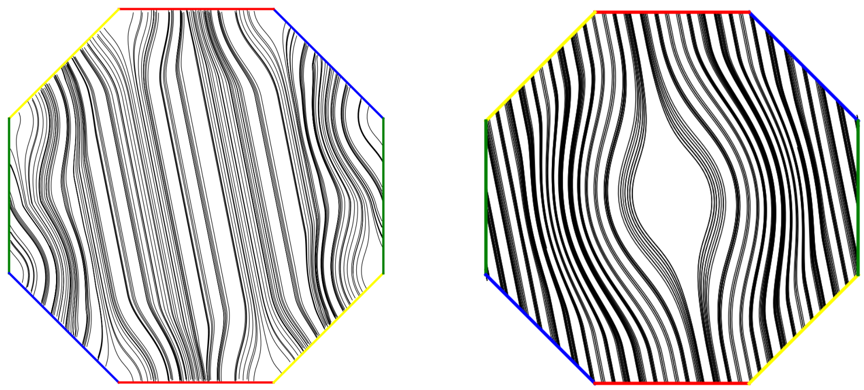


Figure 4. Numerical representations of the set K for a perturbation of a pseudo-Anosov homeomorphism on a genus two surface. Right: perturbation at the unique conical point. Left: perturbation at a regular fixed point.

(iii) The map f is mixing with respect to μ.
(iv) The support of μ is exactly K.

6. The measure μ

In this last section, using extensively Bowen and Ruelle's work [5, 24], we prove that μ is the unique SRB-like measure of f^{-1}, that correlations decrease exponentially fast for $\mathcal{C}^1$ observables compactly supported away from Σ. Finally, using the maximizing property associated with SRB measure, we compute the entropy of f with respect to μ.

We used the term "SRB-like" instead of just "SRB" because SRB measure are only defined for $\mathcal{C}^2$ (or $\mathcal{C}^{1+\alpha}$) diffeomorphisms, but the above map f is only continuous at conical points. Nonetheless, we show that μ is the unique SRB measure associated to $f^{-1}|_{S_g \smallsetminus \Sigma}$ and that the usual definitions of SRB measure extend to f^{-1}. We will therefore refer to SRB measure in the rest of this section instead of "SRB-like" measure.

For now on, we assume that f is a $\mathcal{C}^2$ diffeomorphism away from Σ, which can be achieved by choosing a $\mathcal{C}^2$ bump function k. Such a bump function k is also assumed to be $\mathcal{C}^2$.

6.1. **SRB measure and entropy of f^{-1}.**

Sinai–Ruelle–Bowen measures are particular invariant measures of $\mathcal{C}^2$ transformations. See [28] for a survey about these measures and which dynamical systems have them.

The problem here is that f and f^{-1} are smooth only away from conical points, where they are only continuous. Still, $S_g \smallsetminus \Sigma$ is an invariant set on which f^{-1} is a $\mathcal{C}^2$ diffeomorphism. Furthermore, K is an Axiom A attractor for f^{-1}, in the sense that K is locally maximal, $f^{-1}|_K$ is uniformly hyperbolic and $f^{-1}|_K$ is topologically transitive. Notice that K is connected.

By [24, Theorem 1.5], there exists a unique SRB measure μ_K supported by K, maximizing $h_\nu(f^{-1}|_{S_g\smallsetminus\Sigma}) + \nu(-\log\det \mathrm{d}f^{-1}|_{E^s})$ – and the maximum is equal to 0.

Theorem 6.1. *If W is a curve of finite length contained in $W^{ss}(p)$ and containing p, where p is some hyperbolic fixed point p_i^σ of f, and ν_W is a measure on W with bounded Radon-Nikodym derivative with respect to the measure induce by the Riemann metric on W, then $\mu = \lim_{n\to\infty} (f^{-n})_*\nu_W$.*

In particular, $\mu = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f_*^{-n}\nu_W$ and according to [28], *μ is a SRB measure for $f^{-1}|_{S_g\setminus\Sigma}$. Therefore, by uniqueness, $\mu = \mu_K$.*

Proof. Let $W \subset \tilde{W} \subset W^{ss}(p)$ be a strictly longer curve than W. Let $\tilde{\nu}$ be the measure on $\tilde{W}$ induced by the Riemann metric. By assumption there exists a bounded function $\rho \geqslant 0$ such that $\mathrm{d}\nu_W = \rho\,\mathrm{d}\tilde{\nu}$. If needed, ρ is implicitly extended by 0.

Since k is assumed to be $\mathcal{C}^2$, by Corollary 3.9, h_t is a $\mathcal{C}^1$ flow. Therefore, for small enough t, $(h_t)_*\nu_W$ is supported by $\tilde{W}$ and

$$\mathrm{d}((h_t)_*\nu_W) = \frac{\rho}{\mathrm{Jac}\,h_t}\circ h_{-t}\,\mathrm{d}\tilde{\nu},$$

Where $\mathrm{Jac\,h_t}$ is the Jacobian determinant of the time t of the flow. Therefore, if φ is a continuous function on S_g, then for all small enough t,

$$\begin{aligned}|(h_t)_*(f_*^{-n}\nu_W) - (f_*^{-n}\nu_W)|(\varphi) &= |f_*^{-n}((h_{\lambda^{-n}t})_*\nu_W - \nu_W)|(\varphi)\\ &\leqslant |\varphi|_\infty \int_{\tilde{W}} \left|\frac{\rho}{\mathrm{Jac}\,h_{\lambda^{-n}t}}\circ h_{-\lambda^{-n}t} - \rho\right| \mathrm{d}\tilde{\nu},\end{aligned}$$

which converges, by dominated converge, to zero as n goes to infinity. Therefore, all subsequential limits of $f_*^{-n}\nu_W$ are h_t-invariant. By unique ergodicity of h_t, all subsequential limits of $f_*^{-n}\nu_W$ must coincide with μ. Therefore $f_*^{-n}\nu_W$ converges to μ. □

We can now compute the entropy of f with respect to μ.

Theorem 6.2. *The entropy $h_\mu(f)$ with respect to μ is equal to $\log(\lambda)$.*

Proof. It follows from the fact that $\mathrm{d}f\,v^s = \lambda^{-1}v^s\circ f$ that $\mathrm{d}f^{-1}|_{E^s}$ is constant equal to λ on K. Therefore $h_\mu(f^{-1}|_{S_g\setminus\Sigma}) = \log(\lambda)$. Now, since $\Sigma\cap K = \emptyset$, we get that $h_\mu(f) = h_\mu(f^{-1}) = \log(\lambda)$. □

Finally, remark that since the nonwandering set of f is $K\cap\Sigma$ and since we can extend by continuity $\mathrm{d}f^{-1}|_{E^s}$ at each σ in Σ by $\lambda^{-1}(\lambda+\beta_\sigma) < 1$, the measure mu is still the unique measure maximizing $h_\nu(f^{-1}) + \nu(-\log\det\mathrm{d}f^{-1}|_{E^s})$ for ν ranging over the set of f-invariant measures.

6.2. **Bernoulli and exponential mixing.** Using the careful analysis over Markov partition done by Ruelle in [24], we are able to deduce that (f,μ) is isomorphic to a Bernoulli shift and that the correlations decrease exponentially fast for $\mathcal{C}^1$ observables supported away from Σ.

Theorem 6.3. *The system (f,μ) is isomorphic to a Bernoulli shift.*

Theorem 6.4. *There exist constants $0<\theta<1$ and $C>0$ such that for all $\mathcal{C}^1$ observables φ and ψ compactly supported away from Σ,*

$$|\mu(\varphi\circ f^{-n}\,\psi) - \mu(\varphi)\mu(\psi)| < C||\varphi||_{\mathcal{C}^1}||\psi||_{\mathcal{C}^1}\theta^{-n},\quad \forall n\geqslant 0.$$

The proofs of these two theorems directly follows from [24, Theorem 1.5].

6.3. **What about the Ruelle spectrum?** In [15], Faure, Gouzel and Lanneau proved that for any orientation preserving linear pseudo-Anosov φ map on a surface S_g of genus g, the Ruelle spectrum can be computed explicitly. More precisely, if $\lambda > 1$ is the dilation of φ and λ^{-1}, λ, $\mu_1, \dots, \mu_{2g-2}$ is the spectrum of φ^* – where φ^* is the natural action of φ on the first space of cohomology $H^1(S_g)$ – then the Ruelle spectrum of φ for $\mathcal{C}_c^\infty(S_g \setminus \Sigma)$ observables is $\{\lambda^{-n}\mu_i \mid 1 \leqslant i \leqslant 2g-2,\ n \geqslant 1\}$. Furthermore, the multiplicity of $\lambda^{-n}\mu_i$ is n. In order to prove this result, the authors first show that $\lambda^{-n}\mu_i$ are indeed Ruelle resonances and then that there are no other Ruelle resonances.

Since f is, by construction, homotopic to such linear pseudo-Anosov map φ, the action on the cohomology is the same. One might expect that the Ruelle spectrum of (f, μ) is the same as the one of φ, up to a few modifications.

The key ingredients in the first part of [15] – where it is proved that $\lambda^{-n}\mu_i$ are Ruelle resonances – are the smoothness of the invariant foliations and the uniform contraction of the stable foliation. This particularities remain true in the case of the perturbation f. The argument then should carry over to the case of the specific non-linear pseudo-Anosov maps studied in this paper.

However, the second part of [15] – where it is proved that Ruelle resonances must be of the form $\lambda^{-n}\mu_i$ – relies on many geometric considerations and also on the uniform dilation of the unstable foliation. Unfortunately this last assumption fails, by construction, in the case of f.

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